Midterm 2 solutions

(1) Consider the system

$$\dot{x} = x^2 - y, \qquad \dot{y} = x + y - a,$$

where a is a real parameter.

(a) For what critical value $a = a_c$ does a bifurcation occur?

Solution: Looking for bifurcations we can first look for fixed points. This will occur when $x^2 - y = 0$ and $x + y - a = 0$. Substituting $x^2 = y$ into the second equation gives $x^2 + x - a = 0$. This has solutions

$$x = \frac{-1 \pm \sqrt{1+4a}}{2}.$$

In particular when $a < -1/4$ the term inside the square root is negative and so there are no fixed points and for $a > -1/4$ the term inside the square root is positive and so there will be two fixed points. This reflects a change of behavior and so the bifurcation occurs at $a_c = -1/4$.

(b) Classify the types of fixed points which are present for $a < a_c$ and $a > a_c$. What kind of bifurcation is it?

Solution: As seen in part (a) for $a < a_c$ there are no fixed points. For $a > a_c$ we have that the Jacobian is

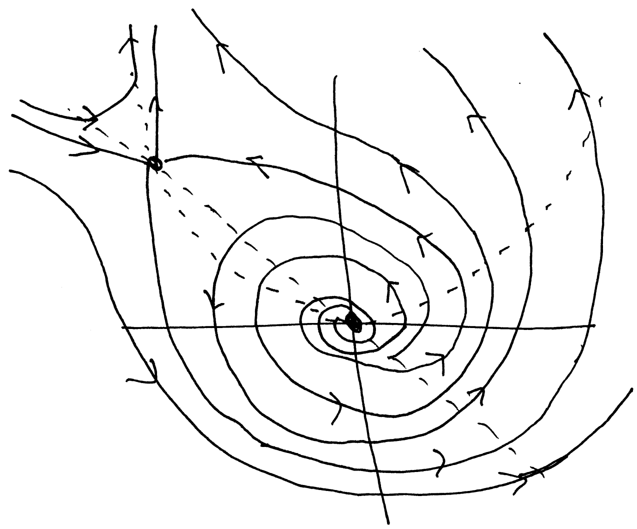
$$\begin{pmatrix} 2x & -1 \\ 1 & 1 \end{pmatrix}.$$

For this matrix both the determinant and trace are $2x + 1$. For $x = \frac{-1}{2} - \frac{\sqrt{1+4a}}{2}$ we have that $2x + 1 = -\sqrt{1+4a} < 0$ which implies that the point is a saddle node. For $x = \frac{-1}{2} + \frac{\sqrt{1+4a}}{2}$ we have that $2x + 1 = \sqrt{1+4a} > 0$ which implies that the point is unstable (possibly a spiral).

Since we go from no fixed points to one saddle and one node this is a saddle-node bifurcation.

(c) Sketch the phase portrait when $a = 0$.

Solution: For $a = 0$ we have that the null clines are $y = x^2$ and $y = -x$. The fixed points are easy to find and are $(-1, 1)$ (which from part (b) is a saddle) and $(0, 0)$ (which from part (b) has eigenvalues satisfying $\lambda^2 - \lambda + 1$ which are complex, i.e., the point is an unstable spiral). A sketch of the phase portrait is shown below.



(2) Consider the system

$$\dot{x} = -y - x(4 - x^2 - y^2), \qquad \dot{y} = x.$$

(a) Use a Liapunov function of the form $V(x, y) = ax^2 + by^2$ to determine whether the origin is asymptotically stable. If so, describe a (fairly large) region which is part of its basin of attraction. (Hint: there are points other than the origin where $\dot{V} = 0$. What happens to a trajectory which reaches such a point?)

SOLUTION: Starting with $V(x, y) = ax^2 + by^2$ we have that

$$\begin{aligned} \dot{V} &= 2ax\dot{x} + 2by\dot{y} \\ &= 2ax\big(-y - x(4 - x^2 - y^2)\big) + 2byx \\ &= xy(2b - 2a) - 2ax^2(4 - x^2 - y^2). \end{aligned}$$

We want to choose a and b so that $V > 0$ except at $(0, 0)$, and so that $\dot{V} < 0$. To eliminate the xy term we should choose $a = b$ and then we can also choose $a > 0$, so let $a = 1$. This then reduces to $\dot{V} = -2x^2(4 - x^2 - y^2)$. Inside the circle $x^2 + y^2 < 4$ this is negative except at $x = 0$. So trajectories will flow to where $x = 0$, at such a point we have $\dot{x} = -y$ and so it will continue to move past that point unless we are at the origin. In particular, the origin's basin of attraction is inside the circle of radius 2 (i.e., $x^2 + y^2 < 4$).

If this seems unsatisfying, we can note that in polar coordinates this system can be rewritten as

$$\begin{aligned} \dot{r} &= \frac{x\dot{x} + y\dot{y}}{r} = \frac{-xy - x^2(4 - x^2 - y^2) + xy}{r} = r\cos^2(\theta)(r^2 - 4) \\ \dot{\theta} &= \frac{x\dot{y} - \dot{x}y}{r^2} = \frac{x^2 + y^2 + xy(4 - x^2 - y^2)}{r^2} = 1 + \frac{1}{2}\sin(2\theta)(4 - r^2) \end{aligned}$$

Now when $\dot{r} = 0$ we either have $r = 0$ (the origin), $r = 2$ (the circle) and when $\cos(\theta) = 0$, in the latter case we have that at those points $\sin(\theta) = 1$ and so these are not fixed points and the trajectories continue rotating. Finally, since when $r < 2$ we have $\dot{r} \leq 0$ and so trajectories move to the origin.

(b) If a trajectory begins at some point $\mathbf{x}$ and does *not* approach the origin, what is its limit set $\omega(\mathbf{x})$? (The answer may be different for different points.)

SOLUTION: On the circle $x^2 + y^2 = 4$ the system reduces to $\dot{x} = -y$ and $\dot{y} = x$. This has a well known solution of $x = -2\cos(t)$ and $y = 2\sin(t)$ and so if we start on the circle we stay on the circle, so in this case the omega limit set is the circle. If we start off the circle we move away from the circle (this can be seen for instance by looking at the polar coordinate expression for $\dot{r}$), in which case the omega limit set is empty.

(3) Consider the system (in polar coordinates)

$$\dot{r} = 6 + 4r\cos\theta - r^2, \qquad \dot{\theta} = 1 + \frac{1}{2}\sin\theta.$$

(**Technically we need to puncture out 0 for this system, the problem is that at 0 there is no well defined trajectory, this is why you will usually see a factor of r in the $\dot{r}$ term so that we do not need to worry about 0. However this fact will make no difference for the remainder of the problem.)

(a) Show that there is a periodic orbit contained in the annulus $1 \leq r \leq 6$.

SOLUTION: On the circle $r = 1$ we have that $\dot{r} = 5 + 4\cos\theta \geq 1$, so that r will increase and move into the annulus. On the circle $r = 6$ we have that $\dot{r} = 24\cos\theta - 30 \leq -6 < 0$ so that r will decrease and again move into the annulus. So the annulus is invariant. Since $\dot{\theta} > 0$ there are also no fixed points in the annulus. It now follows by the Poincaré-Bendixson Theorem a closed orbit exists.

(b) How much thinner could this annulus be and still definitely contain a closed orbit?

SOLUTION: We need our annulus to be invariant, i.e., on the inner side we need to make sure that $\dot{r} \geq 0$ while on the outer side we need to make sure that $\dot{r} \leq 0$. In the first case this reduces to $6 - 4r + r^2 \geq 0$. The maximum such r is the solution to $r^2 + 4r - 6 = 0$ which is at $r = (-4 \pm \sqrt{40})/2 = -2 + \sqrt{10}$ (we can throw out the negative root since that will not work for our problem). Similarly, in the second case this reduces to $6 + 4r - r^2 \leq 0$. The minimum such r is the solution to $r^2 - 4r - 6 = 0$ which is at $r = (4 \pm \sqrt{40})/2 = 2 + \sqrt{10}$ (we can throw out the negative root since that will not work for our problem).

Therefore we can shrink our annulus down to $-2 + \sqrt{10} \leq r \leq 2 + \sqrt{20}$. (This approximates to $1.162 \leq r \leq 5.162$).

(c) Write down an integral which gives the period of the closed orbit. (Do not evaluate it unless you really, really want to).

SOLUTION: To find the period we would integrate $\int_{t_a}^{t_b} dt$. To find this integral we will start with $\dot{\theta} = 1 + \frac{1}{2}\sin\theta$ and rearrange it to $dt = d\theta/(1 + \frac{1}{2}\sin\theta)$. In a full revolution the angles will run from 0 to 2π (any interval of length 2π will work). So we have that

$$T = \int_{t_a}^{t_b} dt = \int_0^{2\pi} \frac{d\theta}{1 + \frac{1}{2}\sin\theta}.$$

By the way this is integrable, but takes a LOT of work. It turns out in this case $T = 4\pi/\sqrt{3} \approx 7.255$.

(4) For the system,

$$\dot{x} = x(y^2 - 2), \qquad \dot{y} = x + y - 3y^3,$$

(a) Use Dulac's criterion with $g(x, y) = x^p$ and a clever choice of the number p to show that there is no closed orbit in the region $x > 0$, and none in the region $x < 0$ either.

SOLUTION: The most clever choice of p is $p = 0$ (i.e., $g = 1$). In this case we have

$$\nabla\cdot(g\dot{\mathbf{x}}) = \frac{\partial}{\partial x}\big(x(y^2 - 2)\big) + \frac{\partial}{\partial y}\big(x + y - 3y^3\big) = (y^2 - 2) + (1 - 9y^2) = -8y^2 - 1 < 0.$$

In particular this is the same sign on the whole plane and so there is no closed orbit anywhere in the plane (also answering part (b)).
Now let us suppose we are a little less clever. Then we want $g(x, y) = x^p$. In this case we have

$$\begin{aligned}\nabla\cdot(g\dot{\mathbf{x}}) &= \frac{\partial}{\partial x}\big(x^{p+1}(y^2 - 2)\big) + \frac{\partial}{\partial y}\big(x^{p+1} + x^p y - 3x^p y^3\big) \\ &= (p + 1)x^p(y^2 - 2) + (x^p - 9x^p y^2) = x^p\big((p - 8)y^2 - (1 + 2p)\big).\end{aligned}$$

From this we see that for any choice of p satisfying $-1/2 \leq p \leq 8$ (and for which x^p makes sense as function) that $\nabla\cdot(g\dot{\mathbf{x}})$ will be negative except when $x = 0$. A good choice for this for instance is $p = 8$ in which case we have $\nabla\cdot(g\dot{\mathbf{x}}) = -17x^8$. In particular, by Dulac's criterion we see that since this is the same sign for $x < 0$ and similarly for $x > 0$ then there can be no closed orbits lying entirely in these regions.

(b) Show that there is no closed orbit anywhere in the xy plane.

SOLUTION: For $x = 0$ our system becomes $\dot{x} = 0$ and $\dot{y} = y - 3y^3$. In particular if we start on the y-axis we stay on the axis and flow to one of the two fixed points which are located at $\pm 1/\sqrt{3}$. We know from part (a) that we cannot lie entirely in $x < 0$ and $x > 0$ but we also know we cannot cross $x = 0$ because we know the trajectories in this case. Therefore no closed orbit exists.