A symmetrical Eulerian identity

Fan Chung* Ron Graham* Don Knuth†

Abstract

We give three proofs for the following symmetrical identity involving binomial coefficients $\binom{n}{m}$ and Eulerian numbers $\left\langle {n \atop m} \right\rangle$:

$$\sum_k \binom{a+b}{k} \left\langle {k \atop a-1} \right\rangle = \sum_k \binom{a+b}{k} \left\langle {k \atop b-1} \right\rangle$$

for any positive integers a and b (where we take $\left\langle {0 \atop 0} \right\rangle = 0$). We also show how this fits into a family of similar (but more complicated) identities for Eulerian numbers.

1 Introduction

Eulerian numbers, introduced by Euler in 1736 [4], while not as ubiquitous as the more familiar Bernoulli numbers, Stirling numbers, harmonic numbers, or binomial coefficients, nevertheless arise in a variety of contexts in enumerative combinatorics. For example, in the enumeration of permutations with a given number of descents [5]. Because the recurrence for Eulerian numbers is a bit more complicated than for many other families of special numbers, and because they increase in size rather rapidly, it was stated in [5] that, "We don't expect the Eulerian numbers to satisfy as many simple identities." Nevertheless, the following identity is rather elegant and appears to be new.

Theorem 1. *For positive integers a and b,*

$$\sum_k \binom{a+b}{k} \left\langle {k \atop a-1} \right\rangle = \sum_k \binom{a+b}{k} \left\langle {k \atop b-1} \right\rangle. \tag{1}$$

We point out here that we will use the convention that the Eulerian number $\left\langle {0 \atop 0} \right\rangle$ is 0 (instead of the more common convention in which this is taken to be 1).

Even for the special case of $b = 1$, the resulting identity is interesting. It states that for any positive integer a,

$$\sum_k \binom{a+1}{k} \left\langle {k \atop a-1} \right\rangle = 2^{a+1} - 1.$$

*University of California, San Diego
†Stanford University

Equation (1) looks superficially like the well known Worpitzky identity [5, 6]

$$x^n = \sum_{k=0}^{n} \binom{x+k}{n} \left\langle {n \atop k} \right\rangle \tag{2}$$

(which converts between two bases for polynomials over $\mathbf{C}$), but is actually quite different. One difference being that the running index of the Eulerian number in (2) is on the bottom, whereas in (1) it is on the top.

In this note, we will give three different proofs of Theorem 1 and also derive some extensions of it. These identities arose in a recent study of descents in permutations which have a restriction on their largest drop [3].

2 A direct proof

We start with the following basic Eulerian identities [5]:

$$\begin{aligned} \left\langle {n \atop m} \right\rangle &= \left\langle {n \atop n-m-1} \right\rangle, \quad \text{for } n \geq 0; \\ \left\langle {n \atop m} \right\rangle &= \sum_{k=0}^{m} (-1)^k \binom{n+1}{k} (m+1-k)^n, \quad \text{for } n > 0. \end{aligned} \tag{3}$$

Lemma 1. *For two positive integers a and b,*

$$\begin{aligned} &\sum_k \binom{a+b}{k} \left\langle {k \atop a-1} \right\rangle \\ &= \sum_{p=-1}^{b-1} \binom{a+b+1}{p+1} (b-p)^{a+b-p-1} (1-b+p)^{p+1} - \sum_{p=0}^{b-1} \binom{a+b}{p} (b-p)^{a+b-p-1} (1-b+p)^p. \end{aligned}$$

Proof.

$$\begin{aligned} \sum_k \binom{a+b}{k} \left\langle {k \atop a-1} \right\rangle &= \sum_k \binom{a+b}{a+b-k} \left\langle {a+b-k \atop a-1} \right\rangle \\ &= \sum_{k=0}^{b} \left(\binom{a+b+1}{a+b-k+1} - \binom{a+b}{a+b-k+1} \right) \left\langle {a+b-k \atop b-k} \right\rangle \\ &= \underbrace{\sum_{k=0}^{b} \binom{a+b+1}{a+b-k+1} \left\langle {a+b-k \atop b-k} \right\rangle}_{=X} - \underbrace{\sum_{k=0}^{b} \binom{a+b}{a+b-k+1} \left\langle {a+b-k \atop b-k} \right\rangle}_{=Y}. \end{aligned}$$

We further expand X using (3):

$$\begin{aligned}
X &= \sum_k \binom{a+b+1}{a+b-k+1} \left\langle {a+b-k \atop b-k} \right\rangle \\
&= \sum_{k=0}^{b} \binom{a+b+1}{a+b-k+1} \sum_{j=0}^{b-k} (-1)^j \binom{a+b-k+1}{j} (b-k+1-j)^{a+b-k} \\
&= \sum_{k=0}^{b} \sum_{j=0}^{b-k} (-1)^j \binom{a+b+1}{k+j} \binom{k+j}{j} (b-k+1-j)^{a+b-k} \\
&= \sum_{p=-1}^{b-1} \sum_{j=0}^{p+1} (-1)^j \binom{a+b+1}{p+1} \binom{p+1}{j} (b-p)^{a+b-p+j-1} \\
&= \sum_{p=-1}^{b-1} \binom{a+b+1}{p+1} (b-p)^{a+b-p-1} \sum_{j=0}^{p+1} (-1)^j \binom{p+1}{j} (b-p)^j \\
&= \sum_{p=-1}^{b-1} \binom{a+b+1}{p+1} (b-p)^{a+b-p-1} (1-b+p)^{p+1}.
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
Y &= \sum_k \binom{a+b}{a+b-k+1} \left\langle {a+b-k \atop b-k} \right\rangle \\
&= \sum_{k=0}^{b} \binom{a+b}{a+b-k+1} \sum_{j=0}^{b-k} (-1)^j \binom{a+b-k+1}{j} (b-k+1-j)^{a+b-k} \\
&= \sum_{k=0}^{b} \sum_{j=0}^{b-k} (-1)^j \binom{a+b}{k+j-1} \binom{k+j-1}{j} (b-k+1-j)^{a+b-k} \\
&= \sum_{p=0}^{b} \binom{a+b}{p} (b-p)^{a+b-p-1} \sum_{j=0}^{p} (-1)^j \binom{p}{j} (b-p)^j \\
&= \sum_{p=0}^{b-1} \binom{a+b}{p} (b-p)^{a+b-p-1} (1-b+p)^p.
\end{aligned}$$

□

Now we will use the following binomial identity of Abel [1]: For $n > 0$,

$$\frac{(x+\alpha)^n}{\alpha} = \sum_{k=0}^{n} \binom{n}{k} (x+k\beta)^{n-k} (\alpha-k\beta)^{k-1}. \tag{4}$$

We use the notation in Lemma 1 with

$$\sum_{k\geq 1} \binom{a+b}{k} \left\langle {k \atop a-1} \right\rangle = X - Y.$$

By using (4), substituting $\alpha = -a, \beta = -1, n = a+b+1, k = a+b-p, x = 1+a$, we have

$$\begin{aligned} X &= \sum_{p=-1}^{b-1} \binom{a+b+1}{p+1} (b-p)^{a+b-p-1} (1-b+p)^{p+1} \\ &= -\frac{1}{a} - \sum_{k=0}^{a} \binom{a+b+1}{a+b-k+1} (k-a)^{k-1} (1+a-k)^{a+b-k+1}. \end{aligned}$$

Also, making the same substitutions as above but with $n = a+b$ in (4), we have

$$\begin{aligned} Y &= \sum_{p=0}^{b-1} \binom{a+b}{p} (b-p)^{a+b-p-1} (1-b+p)^{p} \\ &= -\frac{1}{a} - \sum_{k=0}^{a} \binom{a+b}{a+b-k} (k-a)^{k-1} (1+a-k)^{a+b-k}. \end{aligned}$$

Together, we have

$$\begin{aligned} X - Y &= \sum_{k=0}^{a} (-1)^k \binom{a+b+1}{k} (a-k)^{k-1} (a-k+1)^{a+b-k+1} \\ &\quad - \sum_{k=0}^{a} (-1)^k \binom{a+b}{k} (a-k)^{k-1} (a-k+1)^{a+b-k} \\ &= \sum_{k=0}^{a} (-1)^k \binom{a+b+1}{k} \big((a-k)^k + (a-k)^{k-1}\big) (a-k+1)^{a+b-k} \\ &\quad - \sum_{k=0}^{a} (-1)^k \binom{a+b}{k} (a-k)^{k-1} (a-k+1)^{a+b-k} \\ &= \sum_{k=0}^{a} (-1)^k \binom{a+b+1}{k} (a-k)^{k} (a-k+1)^{a+b-k} \\ &\quad + \sum_{k=1}^{a} (-1)^k \binom{a+b}{k-1} (a-k)^{k-1} (a-k+1)^{a+b-k} \\ &= \sum_{k=-1}^{a-1} (-1)^{k+1} \binom{a+b+1}{k+1} (a-k-1)^{k+1} (a-k)^{a+b-k-1} \\ &\quad - \sum_{k=0}^{a-1} (-1)^k \binom{a+b}{k} (a-k-1)^{k} (a-k)^{a+b-k-1}. \end{aligned}$$

The above expression is exactly equal to

$$\sum_{k \geq 0} \binom{a+b}{k} \left\langle {k \atop b-1} \right\rangle$$

by using Lemma 1 again but interchanging a and b. This completes the first proof of Theorem 1.

3 A bijective proof

We first transform the right-hand side of (1) using the reflection property of Eulerian numbers, and setting $n = a + b$, to

$$\sum_k \binom{n}{k} \left\langle {k \atop a-1} \right\rangle = \sum_k \binom{n}{k} \left\langle {n-k \atop a-k} \right\rangle, \tag{5}$$

Now we interpret the left-hand side (LHS) as the number of strings of length n on the alphabet $\{1, 2, \ldots, a, *\}$ such that each of the pairs $(2, 1)$, $(3, 2)$, $\ldots (a, a-1)$ occurs as a not-necessarily-consecutive substring. For example, one such string when $n = 10$ and $a = 4$ is 3141∗421∗3. A string with k non-$*$ symbols corresponds to one of the permutations of k elements that are enumerated on the LHS; in this case we may regard it as a permutation of $\{0, 1, 2, 3, 5, 6, 7, 9\}$, namely of the indices j in the string $x_0 \ldots x_{n-1}$ where $x_j \neq *$. This permutation is supposed to be one of the $\left\langle {k \atop 3} \right\rangle$ that have exactly 3 descents; indeed, it is 13760925. (First write down the indices j that have $x_j = 1$, then write those with $x_j = 2$, etc.) The general case follows in the same way.

The sum on the right-hand side (RHS) will be nonzero only when $0 \leq k \leq a$. Interpreting it as above, the case $k = 0$ corresponds to strings of length n on $\{1, 2, \ldots, a, a+1\}$ that contain $(2, 1)$, $(3, 2)$, $\ldots$, and $(a+1, a)$. The case $k = 1$ is similar, but on the alphabet $\{1, 2, \ldots a, *\}$. It contains $(2, 1)$, $(3, 2)$, $\ldots$ and $(a, a-1)$ and it must have *exactly one* $*$. The case of general k has alphabet $\{1, \ldots, a+1-k, *\}$, contains $(j+1, j)$ for $1 \leq j < a+1-k$, and has *exactly* k occurrences of $*$.

We now construct a bijection between these two sets of strings.

LHS → RHS: For a string σ in the LHS, let k denote the *least* index (possibly 0) such that either $(k+1, *)$ or (k, k) appears in σ. We map σ to a string τ in the RHS by the following rule:

	$*$	$\rightarrow$	1	
	i	$\rightarrow$	$*$	if $i < k$,
leftmost	k	$\rightarrow$	$*$	
other	k's	$\rightarrow$	1	
	j	$\rightarrow$	$j-k+1$	if $j > k$.

Note that since (i, i) doesn't appear in σ for $i < k$, but i does, then τ has exactly k $*$'s.

RHS → LHS: Let τ be a string in the RHS which has k $*$'s. We first map these $*$'s to the elements $k, k-1, \ldots, 2, 1$ *in order*, with the leftmost $*$ being mapped to k. Then, all 1's to the *left* of the leftmost $*$ get mapped to $*$, and all 1's to the *right* of the leftmost $*$ get mapped to k. Finally, for $2 \leq i \leq a+1-k$, we map i to $i+k-1$. (We recommend that the reader carry out these mappings on a few specific examples to get a feeling for what is happening! For example, in the example mentioned previously with $n = 10, a = 4$, we have 3141∗421∗3 ↔ 422153214 and ∗334324313 ↔ 1∗121∗21∗1.)

To complete the proof, it is now just a matter of checking that these two mappings are indeed a bijection between the LHS and the RHS of (5) (which we leave to the reader) and the proof is complete.

4 A generating function proof

The generating function for our "modified" Eulerian numbers (i.e., with $\left\langle {0 \atop 0} \right\rangle = 0$) is

$$E(w,z) = \frac{e^z - e^{wz}}{e^{wz} - we^z} = \sum_{n,i} \left\langle {n \atop i} \right\rangle w^i \frac{z^n}{n!}. \tag{6}$$

(This is obtained by subtracting 1 from the usual generating function for the Eulerian numbers, which is Eq. (7.56) in [5].)

First, we compute

$$\begin{aligned} e^{wz}E(w,z) &= \sum_k \frac{(wz)^k}{k!} \sum_{n,i} \left\langle {n \atop i} \right\rangle w^i \frac{z^n}{n!} \\ &= \sum_k \frac{w^k z^k}{k!} \sum_{n',i'} \left\langle {n'-k \atop i'-k} \right\rangle w^{i'-k} \frac{z^{n'-k}}{(n'-k)!} \\ &= \sum_{k,n',i'} \frac{1}{k!} \left\langle {n'-k \atop i'-k} \right\rangle w^{i'} \frac{z^{n'}}{(n'-k)!} \\ &= \sum_{k,n,i} \binom{n}{k} \left\langle {n-k \atop i-k} \right\rangle w^i \frac{z^n}{n!}. \end{aligned}$$

Next, we compute

$$\begin{aligned} we^{z}E(w,z) &= w\sum_k \frac{z^k}{k!} \sum_{n,i} \left\langle {n \atop i} \right\rangle w^i \frac{z^n}{n!} \\ &= w\sum_k \frac{z^k}{k!} \sum_{n',i'} \left\langle {n'-k \atop i'-1} \right\rangle w^{i'-1} \frac{z^{n'-k}}{(n'-k)!} \\ &= \sum_{k,n',i'} \frac{1}{k!} \left\langle {n'-k \atop i'-1} \right\rangle w^{i'} \frac{z^{n'}}{(n'-k)!} \\ &= \sum_{k,n,i} \binom{n}{k} \left\langle {n-k \atop i-1} \right\rangle w^i \frac{z^n}{n!}. \end{aligned}$$

But by (6) we have

$$(e^{wz} - we^z)E(w,z) = e^z - e^{wz} = \sum_k \frac{(1-w^k)z^k}{k!}.$$

Hence, by identifying coefficients of $w^i z^n$ in these expressions, we obtain for $n > 0$,

$$\sum_k \binom{n}{k} \left\langle {n-k \atop i-k} \right\rangle - \sum_k \binom{n}{k} \left\langle {n-k \atop i-1} \right\rangle = \begin{cases} 1, & \text{if } i = 0 \neq n, \\ -1, & \text{if } i = n \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

By juggling the variables in (7), (setting $n = a + b$ and $i = a$ while juggling), we can recover (1).

The reason this approach worked was because the multiplier $(e^{wz} - we^{z})$ was divisible by the denominator $e^{wz} - we^{z}$ of $E(w, z)$. We could carry out the same argument with any multiple of $e^{wz} - we^{z}$, for example, $e^{2wz} - w^2 e^{2z}$. In this case, the terms corresponding to the right-hand side of (7) are

$$\begin{aligned}(e^{wz} + we^{z})(e^{z} - e^{wz}) &= e^{(w+1)z} - e^{2wz} + we^{2z} - we^{(w+1)z} \\ &= \sum_k \frac{(w+1)^k z^k}{k!} - \sum_k \frac{2^k w^k z^k}{k!} + w\sum_k \frac{2^k z^k}{k!} - \sum_k w\frac{(w+1)^k z^k}{k!}.\end{aligned}$$

Expanding the corresponding sums and extracting the coefficient of $w^i z^n$ yields for $n, i \geq 0$,

$$\sum_k 2^k \binom{n}{k} \left\langle {n-k \atop i-k} \right\rangle - \sum_k 2^k \binom{n}{k} \left\langle {n-k \atop i-2} \right\rangle = \binom{n}{i} - \binom{n}{i-1} + \begin{cases} 2^n, & \text{if } i = 1 \neq n, \\ -2^n, & \text{if } i = n \neq 1, \\ 0, & \text{otherwise.} \end{cases}$$

More generally, if we use the multiplier $e^{rwz} - w^r e^{rz}$ for a positive integer r, we obtain for $n, i \geq 0$,

$$\sum_k r^k \binom{n}{k} \left\langle {n-k \atop i-k} \right\rangle - \sum_k r^k \binom{n}{k} \left\langle {n-k \atop i-r} \right\rangle = C_r(n,i) + \begin{cases} r^n, & \text{if } i = r-1 \neq n, \\ -r^n, & \text{if } i = n \neq r-1, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$C_r(n,i) = \sum_{j=1}^{r-1} j^{n-i+j-1}(r-j)^{i-j+1}\binom{n}{i-j+1} - \sum_{j=1}^{r-1} j^{n-i+j}(r-j)^{i-j}\binom{n}{i-j}.$$

5 Concluding remarks

A number of questions remain unresolved, some of which we mention here.

1. Can bijective proofs be found for some of the more general identities we have described?
2. Are there interesting identities which would result from taking other multiples of $e^{wz} - we^{z}$?
3. Are there q-analogs to some of these identities?
4. It is well known that $\left\langle {n \atop k} \right\rangle$ also counts the number of permutations π on $[n]$ which have k *drops*, i.e., k elements $i \in [n]$ for which $\pi(i) < i$. With this interpretation, we can replace $\left\langle {n \atop k} \right\rangle$ by $\delta_P(k)$, defined for an arbitrary poset $(P, \prec)$ to be the number of permutations $\pi : P \to P$ which have k drops, which means k elements $x \in P$ such that $\pi(x) \prec x$. With this interpretation, it is sometimes possible to extend results involving Eulerian numbers to this more general setting. For example, such an extension is known for the Worpitsky identity.

   Theorem 2 ([2]). *For a poset $(P, \prec)$ on n points, and any positive integer a, we have*

   $$\sum_k \delta_P(k)\binom{a+k}{n} = \chi_{G(P)}(a) \qquad (8)$$

   where $G(P)$ is the incomparability graph generated by $(P, \prec)$ and $\chi_{G(P)}$ is the chromatic polynomial of $G(P)$.

When $P = [n]$, linearly ordered by size, then $G(P)$ is the empty graph on n vertices and $\chi_{G(P)}(a) = a^n$, so that (8) reduces to the Worpitsky identity (2). Is it possible to extend our results in this direction?

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