The Clique Number of the Graph of Pairwise Sums and Products is 3 or 4

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Abstract

Let $G_+^\times$ be the graph with vertex set $\mathbb{N}$, with n and m adjacent if $n \neq m$ and if there exists positive integers x and y such that $x+y=n$ and $xy=m$ or $x+y=m$ and $xy=n$. The question of whether the chromatic number $\chi(G_+^\times)$ is finite is considered one of the few outstanding problems on partition regularity. We prove that the largest complete subgraph of $G_+^\times$ is either a K_3 or K_4. We prove that there is no K_4 in $G_+^\times$ with two vertices that are consecutive integers, though there are infinitely many K_4 with an edge deleted that have 2 pairs of vertices that are consecutive integers. We also give a 3-coloring of the vertices of $G_+^\times$ such that no triangle in $G_+^\times$ has all its vertices of the same color.

1 Introduction

In 1916, while trying to prove Fermat's Last Theorem, Schur [14] proved what is arguably the first result in Ramsey theory . Schur's theorem states that for every positive integer t, there exists a least positive integer $S(t)$ such that for every t-coloring of the positive integers from 1 to $S(t)$ there is a, b, and x of one color with $a+b=x$. As noted by Hindman et al. [11], it follows from Schur's theorem that for every positive integer t, there exists a least positive integer $P(t)$ such that for every t-coloring of the positive integers from 2 to $P(t)$ there is a, b, and y of one color with $ab=y$.

It follows that for any finite coloring of the positive integers, there exist a and b such that a, b, and $a+b$ are monochromatic. Likewise, for any finite coloring of the positive integers, there exist a and b such that a, b, and ab are monochromatic. Hindman has asked, in several different articles, whether a stronger result involving both sums and products holds [6], [8], [9], [10], [11].

Problem 1 *For every finite coloring of the positive integers, does there exist a and b such that a, b, $a+b$, and ab are all the same color?*

This problem has been open for at least 25 years, with few results supporting conclusions in either direction. Graham [8] proved that for every 2-coloring of the integers $[1, 252]$, there exist a and b such that a, b, $a+b$, and ab are all the same color, and that 252 is the smallest positive integer for which this is true.

Hindman et al. [11] state that "it is rather extraordinary" that the following weaker version of Problem 1 has not even been resolved.

Problem 2 *For every finite coloring of the positive integers, does there exist a and b, not both equal to 2, such that $a+b$ and ab are both the same color?*

We do not allow both a and b to be 2 because this would trivialize the problem. In investigating this problem, it is natural to define the following graph: let $G_+^\times$ be the graph with vertex set $\mathbb{N}$, such that vertices x and y, not both equal to 2, are adjacent if there exist positive integers a and b such that $a+b=x$ and $ab=y$.

The *chromatic number* $\chi(G_+^\times)$ is the smallest positive integer r for which there is an r-coloring of the positive integers such that for every pair of distinct positive integers a and b not both equal to 2, the numbers ab and $a+b$ are not the same color. Therefore, Problem 2 is equivalent to determining whether or not $\chi(G_+^\times)=\infty$. Halbeisen [7] recently showed that $\chi(G_+^\times) \geq 4$ by exhibiting a subgraph of $G_+^\times$ with chromatic number 4.

In Section 2, we prove inequalities concerning the comparative size of the largest and smallest vertices in every triangle in $G_+^\times$. In Section 3, we prove that $G_+^\times$ does not have a K_5 subgraph. Because we know that $G_+^\times$ contains triangles, such as the one on vertices 6, 7, 8, it follows that the largest complete subgraph of $G_+^\times$ is either K_3 or K_4. Using stronger results concerning triangles of $G_+^\times$ with two vertices that are consecutive positive integers, we prove that $G_+^\times$ does not have a K_4 subgraph with 2 of its vertices being consecutive integers, though there are infinitely many subgraphs isomorphic to K_4 with an edge deleted whose vertices consist of two pairs of consecutive integers. Finally, we give a 3-coloring of the positive integers such that no three vertices of a triangle in $G_+^\times$ are all the same color.

2 Inequalities on Triangles in $G_+^\times$

The following lemma gives basic bounds on adjacent vertices of $G_+^\times$.

Lemma 1 *If a and b are positive integers and $x=a+b$ and $y=ab$ are adjacent vertices of $G_+^\times$, then*

$$x-1 \leq y \leq \frac{x^2}{4}.$$

Also, if neither a nor b is equal to 1, then $y>x$.

Proof: We have

$$\frac{x^2}{4} - y = \frac{(a+b)^2}{4} - ab = \frac{(a-b)^2}{4} \geq 0,$$

so $y \leq \frac{x^2}{4}$. We may assume without loss of generality that $a \leq b$. If $a = 1$, then $y = x - 1$. If $a = 2$, then $b \geq 3$ and $x = a + b < 2b = ab = y$. If $a > 2$, then $x = a + b \leq 2b < ab = y$. $\square$

The following lemma will be very useful in the proofs of several later lemmas.

Lemma 2 *Let x be an integer and a and b be distinct positive integers with $a + b < x$. We have $a < b$ if and only if*

$$a(x-a) < b(x-b).$$

Proof: Assume $a < b$. Since $a + b < x$, then $(a+b)(b-a) < x(b-a)$. Rearranging, we have $a(x-a) < b(x-b)$.

Assume $a(x-a) < b(x-b)$ and for contradiction, that $a \geq b$. We can rewrite $a(x-a) < b(x-b)$ as $(b-a)(a+b) < x(b-a)$, so $a + b > x$ which contradicts the assumption that $a + b < x$. $\square$

We now show that there are no triangles in $G_+^\times$ with vertices x, y, and z, no two of which are consecutive integers, with x, y, and z relatively "close" together.

Theorem 1 *If x, y, and z are vertices of a triangle in $G_+^\times$ with $x+1 < y < z-1$, then*

$$\frac{x(x+6)^{\frac{1}{2}}}{2} \leq z \leq \frac{x^2}{4}$$

Proof: From Lemma 1, we have $z \leq \frac{x^2}{4}$.

Since no two of x, y, and z are consecutive, it follows that the larger of each pair of adjacent vertices is the product and the smaller is the sum. Therefore, there are positive integers a, b, and c such that
$x = a + (x-a)$ and $y = a(x-a)$ with $1 < a \leq x - a < x$,
$x = b + (x-b)$ and $z = b(x-b)$ with $1 < b \leq x - b < x$, and
$y = c + (y-c)$ and $z = c(y-c)$ with $1 < c \leq y - c < y$.
Combining these equations,

$$b(x-b) = z = c(a(x-a) - c).$$

Simplifying,

$$x(ca - b) = ca^2 + c^2 - b^2. \tag{1}$$

Since $a(x-a) = y < z = b(x-b)$, we know that $a \neq b$. Moreover, since $a \leq x - a$ and $b \leq x - b$, we have $a + b < x$. By Lemma 2 we conclude that $a < b$.

Since $y > x$, we have $b(y-b) > b(x-b) = z = c(y-c)$. Also, since $b \leq x-b$, then $b \leq \frac{x}{2} < \frac{y}{2}$. Since $c \leq y-c$, we have $c \leq \frac{y}{2}$. Thus $b+c < y$. Therefore, by Lemma 2, we have $c < b$.

Case 1: $b = ca$. In this case, we have from (1) that $0 = ca^2 + c^2 - b^2 = ca^2 + c^2 - (ca)^2$. Since $c > 0$, we may divide out by c and solve for c:

$$c = 1 + \frac{1}{a^2 - 1}$$

Since c is a positive integer, $a^2 - 1$ must be a factor of 1, so $a^2 = 0$ or 2, which contradicts a being a positive integer.

Case 2: $b \neq ca$. We solve for x in (1):

$$x = \frac{ca^2 + c^2 - b^2}{ca - b} = a + \frac{ba + c^2 - b^2}{ca - b}$$

Case 2a: $b > ca$. In this case, since a, b, $c \geq 2$ and $b - ca \geq 1$, then

$$x = a + \frac{ba + c^2 - b^2}{ca - b} = a + \frac{-ba - c^2 + b^2}{b - ca}$$

$$\leq a - ba - c^2 + b^2 = b^2 + a(1-b) - c^2$$

$$\leq b^2 - 2b - 2 \leq b^2 - 6.$$

Case 2b: $b < ca$. In this case, since $a \leq b-1$, $c \leq b-1$, $b \geq 3$, and $ca - b \geq 1$, then

$$x = a + \frac{ba + c^2 - b^2}{ca - b} \leq a + ba + c^2 - b^2$$

$$\leq (b-1) + b(b-1) + (b-1)^2 - b^2 = b^2 - 2b \leq b^2 - 6.$$

In both Case 2a and 2b, $(x+6)^{\frac{1}{2}} \leq b$ and

$$z = b(x-b) \geq b(x - \frac{x}{2}) = \frac{bx}{2} \geq \frac{x(x+6)^{\frac{1}{2}}}{2}.$$

□

Theorem 1 gives lower and upper bounds on the largest vertex in terms of the smallest vertex in a triangle when no two of the vertices are consecutive integers. Lemmas 3 and 4 give analogous bounds in the cases not handled by Theorem 1.

Lemma 3 *If $x, x+1, y$ are vertices of a triangle in $G_+^\times$ and $x+1 < y$, then*

$$\frac{x^2 + 2x}{6} \leq y \leq \frac{x^2}{4}$$

Proof: Case 1: $y = x + 2$. In this case, there exists a with $1 < a \leq x - a < x$ such that $a(x - a) = y = x + 2$. But since $a \geq 2$, then by Lemma 2 we have $2(x - 2) \leq a(x - a) \leq x + 2$. Therefore, $x \leq 6$. The only such triangle is 6, 7, 8. In that case, $x = 6$ and $y = 8$, and we see that $\frac{6^2+12}{6} = 8 \leq y = 8 < 9 = \frac{6^2}{4}$.

Case 2: $y > x + 2$. In this case, there exist a and b with $a(x - a) = y = b(x + 1 - b)$, $1 < a \leq x - a < x$, and $1 < b \leq x + 1 - b < x + 1$. From Lemma 1, we have $y \leq \frac{x^2}{4}$. Since $b \leq \frac{x+1}{2}$ and $a \leq \frac{x}{2}$, then $a + b < x + 1$. Since $a + b < x + 1$ and $a(x + 1 - a) > a(x - a) = b(x + 1 - b)$, then by Lemma 2 we have that $a > b$. Solving the equation $a(x - a) = b(x + 1 - b)$ for x, we have $x = \frac{b}{a-b} + a + b \leq b + a + b \leq 3a - 2$. Therefore, $\frac{x+2}{3} \leq a \leq \frac{x}{2}$. Therefore, we have

$$y = a(x - a) \geq a(x - \frac{x}{2}) = \frac{ax}{2} \geq \frac{x^2 + 2x}{6}$$

□

Lemma 4 *The positive integers $x, y-1, y$ are vertices of a triangle in $G_+^\times$ with $x < y - 1$ if and only if $x = 6$ and $y = 8$, or $x \geq 6$ is even and $y = \frac{x^2}{4}$.*

Proof: As in Case 1 of Lemma 3, if $x = y - 2$ then the only triangle with all three vertices consecutive integers has vertices 6, 7, 8, so $x = 6$ and $y = 8$ in this case.

If $x < y - 2$, then there exist distinct positive integers a and b such that $y - 1 = a(x - a)$, $y = b(x - b)$, $1 < a \leq x - a < x$, and $1 < b \leq x - b < x$. Since $b(x - b) = a(x - a) + 1 > a(x - a)$ and $a + b < x$, then by Lemma 2 we have that $b > a$. We can rearrange the equation $b(x - b) = a(x - a) + 1$ as

$$(b - a)(x - (b + a)) = 1$$

Since $b - a$ is a positive factor of 1, then $b - a = 1$. So $x - (b + a) = 1$, and substituting for a, we get $b = \frac{x}{2}$. Therefore, $y = b(x - b) = \frac{x^2}{4}$. If x were odd, then y would not be an integer. Therefore, x must be even.

If $x \geq 6$ is even and $y = \frac{x^2}{4}$, then x, $y - 1$, and y are the vertices of triangle:

$$x = \frac{x}{2} + \frac{x}{2} = (\frac{x}{2} - 1) + (\frac{x}{2} + 1), \ y - 1 = (\frac{x}{2} - 1)(\frac{x}{2} + 1), \text{ and } y = (\frac{x}{2})(\frac{x}{2}).$$

The vertices $y - 1$ and y are trivially adjacent. □

3 Main Results

We will use the following theorem in the proof that $G_+^\times$ does not contain a K_5 subgraph.

Theorem 2 *There are no K_4 subgraphs of $G_+^\times$ that contain two consecutive integers as vertices.*

Proof: Assume x, $x+1$, y, z are vertices of K_4 in $G_+^\times$. Without loss of generality, $z > y$.

Case 1: $y < x$. In this case, by Lemma 4, either $y = 6$ and $x+1 = 8$, or $y \geq 6$ and $x+1 = \frac{y^2}{4}$.

Case 1a: $y = 6$ and $x+1 = 8$. Since $z > y$, then $z > 6$. By Lemma 1, $z \leq \frac{y^2}{4} = 9$. So $z = 9$, but $x = 7$ and $z = 9$ are not adjacent.

Case 1b: $y \geq 6$ and $x+1 = \frac{y^2}{4}$. If $z > x+1$, then y and z can't be adjacent by Lemma 1. So $z < x$, and applying Lemma 4, $z = y$ or $z = 6 \leq y$, which both contradict $z > y$.

Case 2: $z > y > x+1 > x$. By Lemma 3, $\frac{x^2+2x}{6} \leq y \leq \frac{x^2}{4}$ and $\frac{x^2+2x}{6} \leq z \leq \frac{x^2}{4}$. If $z = y+1$, then applying Lemma 4, x and $x+1$ would have to be equal, a contradiction. Thus $z > y+1$, and there exists a positive integer $a > 1$ for which $z = a(y-a)$. By Lemma 3,

$$\frac{x^2}{4} \geq z = a(y-a) \geq 2(y-2) = 2y - 4 \geq 2(\frac{x^2+2x}{6}) - 4.$$

The inequality $\frac{x^2}{4} \geq 2(\frac{x^2+2x}{6}) - 4$ fails for $x > 4$, but none of 1, 2, 3, or 4 are vertices of a triangle in $G_+^\times$. Hence, there are no cliques with 4 vertices that contain two consecutive integers. ☐

For convenience, we let $f(x) = \frac{x(x+6)^{\frac{1}{2}}}{2}$.

If x_1, x_2, x_3, x_4, and x_5, written in increasing order, are the vertices of a K_5 in $G_+^\times$, then $x_{i+1} > x_i + 1$ for $1 \leq i \leq 4$ by Theorem 2. We now can apply Theorem 1 to show that no K_5 are in $G_+^\times$, as follows.

Theorem 3 *The graph $G_+^\times$ does not have a K_5 subgraph.*

Proof: Assume x_1, x_2, x_3, x_4, and x_5 are the vertices of a K_5 in $G_+^\times$, written in increasing order. Since x_5 is adjacent to x_1, then $x_5 \leq \frac{x_1^2}{4}$. Applying Theorem 1 to the triangle with vertices x_1, x_2, and x_3, we have $x_3 \geq f(x_1)$. Applying Theorem 1 to the triangle with vertices x_3, x_4, and x_5, we have

$$\frac{x_1^2}{4} \geq x_5 \geq f(x_3) \geq f(f(x_1))$$

However, the inequality $\frac{x_1^2}{4} \geq f(f(x_1))$ fails for $x_1 \geq 1$. ☐

Theorem 4 *There is a 3-coloring of the vertices of $G_+^\times$ such that no triangle of $G_+^\times$ has all 3 vertices the same color.*

Proof: Let $x_1 = 7$, and for $i \geq 1$, let $x_{i+1} = \lfloor f(x_i) \rfloor$. We first partition the integers into intervals:

Let $I_1 = [1, x_1)$, and for $i > 1$, let $I_i = [x_{i-1}, x_i)$.

If $i \equiv 0 \pmod 3$, let every positive integer in I_i be colored green.

If $i \equiv 1 \pmod 3$, let every positive integer in I_i be colored red.

If $i \equiv 2 \pmod 3$, let every positive integer in I_i be colored blue.

We now show that there are no triangles in this 3-coloring whose vertices are the same color. Assume for contradiction that there are monochromatic x, y, and z that are the vertices of a triangle in $G_{+}^{\times}$. Without loss of generality, we may assume $x < y < z$. It is clear from a quick check that none of the integers in I_1 are in a K_3 with red vertices. If x, y, $z \in I_i$ with $i > 1$, then $z < f(x)$, and so from Theorem 1, Lemma 3, and Lemma 4, they do not form a triangle. If $x \in I_i$ and $z \notin I_i$, then $z \in I_{i+3n}$ for some positive integer n. So $x_i > x$, and $z \geq x_{i+2}$. By definition, $x_{i+2} = \lfloor f(\lfloor f(x_i) \rfloor) \rfloor$. Since $x_i \geq 12$ for $i \geq 2$, it follows that $z \geq x_{i+2} > \frac{x_i^2}{4} \geq \frac{x^2}{4}$. But x and z are adjacent, which contradicts Lemma 1. □

Theorem 5 *There exists an infinite family* $\{x_1, x_1+1, x_2, x_2+1\}$ *of sets of 4 distinct positive integers such that the subgraph induced by these vertices in* $G_{+}^{\times}$ *is* K_4 *with one edge deleted. Namely, for every integer* $c \geq 2$, *we may take* $x_1 = (c^2+c-1)^2 - 1$ *and* $x_2 = 2c^2 + 2c - 2$.

Proof: It is clear that x_1 and x_1+1 are adjacent and that x_2 and x_2+1 are adjacent. Also,
$x_2 = (c^2+c-1) + (c^2+c-1) = (c^2+c) + (c^2+c-2)$,
$x_2 + 1 = (c^2-1) + (c^2+2c)$,
$x_1 = (c^2+c)(c^2+c-2) = (c^2-1)(c^2+2c)$, and
$x_1 + 1 = (c^2+c-1)(c^2+c-1)$.
This shows that all these vertices are pairwise adjacent, except for x_1+1 and x_2+1. Since Theorem 2 shows that there are no K_4 in $G_{+}^{\times}$ with two of the vertices being consecutive integers, x_1+1 and x_2+1 are not adjacent. □

4 Conclusion

Similar to the problem of determining the chromatic number of $G_{+}^{\times}$, we think that determining the subgraphs of $G_{+}^{\times}$ is also an interesting problem.

With the help of a computer, we have shown that there are no K_4 subgraphs of $G_{+}^{\times}$ whose minimum vertex is at most $1,000$. This computer evidence supports the following conjecture.

Conjecture 1 *The graph* $G_{+}^{\times}$ *has no* K_4 *subgraph.*

Let $F(n)$ be the maximum number of edges over all subgraphs of $G_{+}^{\times}$ with n vertices. Since every graph has at most $\binom{n}{2}$ edges, then the obvious upper bound on $F(n)$ is $F(n) \leq \binom{n}{2}$. Turan's famous extremal graph theorem [1] says that the number of edges of every K_p-free graph on n vertices is at most $\frac{p-2}{2p-2}n^2$. Since we have shown that no K_5 occurs, then we have the improved upper bound $F(n) \leq \frac{3}{8}n^2$. By considering the subgraph of $G_{+}^{\times}$ induced by the first n positive integers we get the lower bound $F(n) \geq (\frac{1}{2} + o(1))n \ln n$. We summarize the best known bounds on $F(n)$ in Lemma 5.

Lemma 5 *For all positive integers n, we have*

$$(\frac{1}{2} + o(1))n \ln n \leq F(n) \leq \frac{3}{8}n^2.$$

From Lemma 5 we see that the lower and upper bounds on $F(n)$ are far apart. This leads us to the following problem.

Problem 3 *Determine whether or not $F(n) = o(n^2)$.*

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