Infinite color analogue of Rado's theorem

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Abstract

Let R be a subring of the complex numbers and $\mathfrak{a}$ be a cardinal. A system of linear homogeneous equations with coefficients in R is called $\mathfrak{a}$-*regular* over R if for every $\mathfrak{a}$-coloring of R there is a monochromatic solution to that system in distinct variables. Rado in 1943 classified those systems of linear homogeneous equations that are $\mathfrak{a}$-regular over R for all positive integers $\mathfrak{a}$. For every infinite cardinal $\mathfrak{a}$, we classify those systems of linear homogeneous equations that are $\mathfrak{a}$-regular over R. As a corollary, for every positive integer s, we have $2^{\aleph_0} > \aleph_s$ if and only if the equation $x_0 + sx_1 = x_2 + \cdots + x_{s+2}$ is $\aleph_0$-regular over $\mathbb{R}$. The case $s = 1$ is due to Erdős and Kakutani.

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1 Introduction

In 1916, Schur [20] proved that for every finite coloring of the positive integers, there are distinct positive integers x, y, and z that are all the same color and satisfy $x + y = z$. In 1927, van der Waerden [23] proved his celebrated theorem that every finite coloring of the positive integers contains arbitrarily long monochromatic arithmetic progressions. These two classical theorems of Schur and van der Waerden were beautifully generalized by Rado in his 1933 thesis [17] and even further in 1943 [18].

For an $m \times n$ matrix $A = (a_{ij})$ with coefficients in a subring R of the complex numbers, denote by $\mathcal{L} = \mathcal{L}(A)$ the system of linear homogeneous equations

$$\sum_{j=1}^{n} a_{ij}x_j = 0 \qquad \text{for } 1 \leq i \leq m. \tag{1}$$

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For a cardinal $\mathfrak{a}$, the system $\mathcal{L}$ is called $\mathfrak{a}$-*regular* over R if for every $\mathfrak{a}$-coloring of R there is a monochromatic solution to $\mathcal{L}$ in distinct variables. The system $\mathcal{L}$ is called *regular* over R if it is $\mathfrak{a}$-regular over R for all positive integers $\mathfrak{a}$. The matrix A with column vectors $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$ is said to satisfy the *columns condition* if there exists a partition $\{1, 2, \ldots, n\} = \bigcup_{j=1}^{p} S_j$ such that $\sum_{i \in S_1} \mathbf{c_i} = \mathbf{0}$ and for each $j \in \{2, 3, \ldots, p\}$, $\sum_{i \in S_j} \mathbf{c_i}$ is a linear combination of $\{\mathbf{c_i} : i \in \bigcup_{k=1}^{j-1} S_k\}$. Rado proved in 1943 that the system $\mathcal{L}(A)$ is regular over R if and only if the matrix A satisfies the columns condition and there is a solution to $\mathcal{L}(A)$ in distinct variables.

For R a subring of the complex numbers and infinite cardinal $\mathfrak{a}$, we classify in Theorem 1 those systems of linear homogeneous equations that are $\mathfrak{a}$-regular over R. Following Komjáth [16], we define a partition $\mathcal{P} = \bigcup_{i=1}^{l} P_i$ of the set $\mathcal{P} = \{1, \ldots, n\}$ to be *balanced* if $\sum_{j \in P_k} a_{i,j} = 0$ holds for every i and k with $1 \leq i \leq m$ and $1 \leq k \leq l$. A collection $\{\mathcal{P}_1, \ldots, \mathcal{P}_s\}$ of balanced partitions is called *separative* if for all $x, y \in \{1, 2, \ldots, n\}$ with $x \neq y$, there is a balanced partition $\mathcal{P}_j$ with x and y in different sets in $\mathcal{P}_j$. We call $\mathcal{L}$ *separable* if there exists a separative collection of balanced partitions of $\mathcal{L}$. For separable $\mathcal{L}$, the separation number $s(\mathcal{L})$ is the least positive integer s such that there exists a separative collection of s balanced partitions.

In Section 4 we prove Theorem 1, which extends Rado's theorem to infinite colorings.

Theorem 1 *For an infinite cardinal $\mathfrak{a}$ and subring R of the complex numbers, a system $\mathcal{L}$ of linear homogeneous equations with coefficients in R is $\mathfrak{a}$-regular over R if and only if $\mathfrak{L}$ is separable and $|R| \geq \mathfrak{a}^{+s(\mathcal{L})}$. Moreover, for any countable subfield $F \subset R$, there is an $\mathfrak{a}$-coloring of R that is free of monochromatic solutions to all systems $\mathcal{L}$ of linear homogeneous equations such that the coefficients of $\mathcal{L}$ are in F and $\mathcal{L}$ is not $\mathfrak{a}$-regular over R.*

Several special cases of Theorem 1 have already been solved. Erdős and Kakutani [10] in 1943 proved that the $2^{\aleph_0} = \aleph_1$ if and only if there is a countable coloring of the real numbers such that each monochromatic subset is linearly independent over $\mathbb{Q}$. Erdős [16] followed this by proving that $x_1 + x_2 = x_3 + x_4$ is $\aleph_0$-regular over $\mathbb{R}$ if and only if $2^{\aleph_0} \geq \aleph_2$. A proof of this result can be found in Davies [5]. According to Komjáth [16], Rado observed a long time ago that every $\mathbb{Q}$-vector space has a countable coloring without any monochromatic solutions to $x_1 + x_2 = 2x_3$ with x_1, x_2, and x_3 distinct. In 1969, Ceder [3] made the same observation. Later that year, Ceder [4] showed that there are no linear homogeneous equations in 3 variables that are $\aleph_0$-regular over $\mathbb{R}$. This follows from Theorem 1 since no linear homogeneous equation in 3 variables is separable.

For each positive integer s, Komjáth [16] gave an example of a system of linear homogeneous equations that is $\aleph_0$-regular over $\mathbb{R}$ if and only if $2^{\aleph_0} > \aleph_s$. Likewise, for each positive integer s, Corollary 1 gives an example of a single linear homogeneous equation that is $\aleph_0$-regular over $\mathbb{R}$ if and only if $2^{\aleph_0} > \aleph_s$. Corollary 1 follows from Theorem 1 since the separation number for equation (2) is $s+1$. Corollary 1 generalizes the case $s = 1$ that Erdős [16] solved.

Corollary 1 *For every positive integer s, the linear homogeneous equation*

$$x_1 + sx_2 = x_3 + \cdots + x_{s+3} \tag{2}$$

is $\aleph_0$*-regular over* $\mathbb{R}$ *if and only if* $2^{\aleph_0} > \aleph_s$.

For a system $\mathcal{L}$ of linear homogeneous equations with rational coefficients, Komjáth defines $\lambda(\mathcal{L})$ to be the least cardinal $\mathfrak{b}$ such that if V is a rational vector space of dimension $\mathfrak{b}$, then every countable coloring of V has a monochromatic solution to $\mathfrak{L}$ in distinct variables. If no such cardinal $\mathfrak{b}$ exists, set $\lambda(\mathcal{L}) = \infty$. We note that the dimension and cardinality of an uncountable rational vector space are equal. Komjáth proved that if $\lambda(\mathcal{L}) \leq 2^{\aleph_0}$, then $\mathcal{L}$ is separable. He also proved that if $\mathcal{L}$ is separable, then $\lambda(\mathcal{L}) \leq \aleph_{s(\mathcal{L})}$, where $s(\mathcal{L})$ is the separation number of $\mathcal{L}$. The following theorem demonstrates that Komjáth's upper bound on $\lambda(\mathcal{L})$ is tight.

Theorem 2 *If* $\aleph_s \leq 2^{\aleph_0}$*, then a system* $\mathcal{L}$ *of linear homogeneous equations with rational coefficients satisfies* $\lambda(\mathcal{L}) = \aleph_s$ *if and only if* $\mathcal{L}$ *is separable and* $s = s(\mathcal{L})$*.*

We deduce Theorem 1 and Theorem 2 from a result concerning hypergraph Ramsey numbers. To do this, we first need some terminology. A *hypergraph* $H = (V, E)$ consists of a set V and a collection E of subsets of V. The elements of V are called *vertices* and the elements of E are called *edges*. A hypergraph H is called k*-uniform* if every edge contains exactly k vertices.

Ramsey's theorem states that for positive integers r and k, every r-coloring of the edges of an infinite complete k-uniform hypergraph contains an infinite monochromatic complete k-uniform hypergraph. The Erdős–Rado arrow notation $\mathfrak{b} \rightarrow (\mathfrak{d})^k_{\mathfrak{a}}$ means that for every $\mathfrak{a}$-coloring of the subsets of cardinality k of a set of cardinality $\mathfrak{b}$, there is a monochromatic complete k-uniform hypergraph on $\mathfrak{d}$ vertices. Ramsey's theorem can be written using the Rado arrow notation:

$$\aleph_0 \rightarrow (\aleph_0)^k_r.$$

The successor cardinal of $\mathfrak{a}$ is denoted by $\mathfrak{a}^+$, and the s^{th} successor cardinal of $\mathfrak{a}$ is denoted by $\mathfrak{a}^{+s}$. Erdős [8], [11] in 1942 proved if $\mathfrak{a} \geq \aleph_0$, then

$$(2^{\mathfrak{a}})^+ \rightarrow (\mathfrak{a}^+)^2_{\mathfrak{a}}, \tag{3}$$

and

$$2^{\mathfrak{a}} \not\rightarrow (3)^2_{\mathfrak{a}}. \tag{4}$$

In 1943, Erdős and Kakutani [10] proved that the edges of the complete graph on $\aleph_1$ vertices can be countably colored without any monochromatic cycles, but that every countable coloring of the edges of the complete graph on $\aleph_2$ vertices contains a monochromatic cycle. For $\mathcal{G}$ a nonempty family of k-uniform hypergraphs and cardinals $\mathfrak{a}$ and $\mathfrak{b}$, the partition relation $\mathfrak{b} \rightarrow (\mathcal{G})^k_{\mathfrak{a}}$ is said to hold if for every $\mathfrak{a}$-coloring of the edges of the complete k-uniform hypergraph $K^{(k)}_{\mathfrak{b}}$ on $\mathfrak{b}$ vertices, there is a monochromatic copy of a hypergraph $G \in \mathcal{G}$. For a nonempty family $\mathcal{G}$ of k-uniform hypergraphs and cardinal $\mathfrak{a}$, the *Ramsey number* $R(\mathcal{G}, \mathfrak{a})$ is the least cardinal $\mathfrak{b}$ such that the partition relation $\mathfrak{b} \rightarrow (\mathcal{G})^k_{\mathfrak{a}}$ holds. The Erdős-Kakutani result mentioned above can be restated as saying that for the family $\mathcal{G} = \{C_i\}_{i \geq 3}$ of cycles, we have $R(\mathcal{G}, \aleph_0) = \aleph_2$.

The author [12] recently determined $R(\mathcal{G}, \mathfrak{a})$ for every nonempty family $\mathcal{G}$ of finite graphs and infinite cardinal $\mathfrak{a}$. This is Theorem 3 below. A star S_n is a graph on $n+1$ vertices with one vertex of degree n and the other n vertices of degree 1. A *galaxy* is a graph whose connected components are stars.

Theorem 3 *(Fox, [12]) Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinals with $\mathfrak{a}$ infinite. For a nonempty family $\mathcal{G}$ of finite graphs, the partition relation $\mathfrak{b} \to (\mathcal{G})^2_{\mathfrak{a}}$ holds if and only if (1), (2), (3), or (4) below are true.*

(1) There exists $G = (V, E) \in \mathcal{G}$ with $|E| \leq 1$ and $|V| \leq \mathfrak{b}$.
(2) $\mathfrak{b} = \mathfrak{a}^+$ and there exists a galaxy $G \in \mathcal{G}$.
(3) $\mathfrak{b} > \mathfrak{a}^+$ and there exists a bipartite graph $G \in \mathcal{G}$.
(4) $\mathfrak{b} > 2^{\mathfrak{a}}$.

Theorem 3 clearly strengthens the Erdős–Kakutani result, as Theorem 3 implies that there is a countable coloring of the edges of the complete graph on $\aleph_1$ vertices such that the only connected monochromatic subgraphs are stars. The forward directions of *(3)* and *(4)* of Theorem 3 follow from the Erdős-Hajnal partition relation (5) below with $k = 2$ and the Erdős partition relation (3), respectively.

A k-uniform hypergraph $H = (V, E)$ is called *k-partite* if there exists a partition $V = V_1 \cup \ldots \cup V_k$ of the vertex set V such that every edge of H contains exactly one vertex in each V_i. If H is k-partite for some positive integer k, then we simply call H *partite*. We call a hypergraph $H = (V, E)$ an *s-hybrid* if there exists a positive integer k and a partition of the vertex set $V = V_1 \cup \ldots \cup V_k$ such that every every edge of H contains exactly one vertex in each V_i, and for each s-tuple $(v_1, \ldots, v_s)$ with $v_i \in V_i$, there is at most one edge $e \in E$ such that $\{v_1, \ldots, v_s\} \subset e$. Notice that if $s > k$, then a k-partite hypergraph is vacuously an s-hybrid. In particular, a graph is a 1-hybrid if and only if it is a galaxy. From the definitions, an s-hybrid hypergraph is necessarily partite, and a k-partite hypergraph is necessarily k-uniform.

A polarized partition theorem due to Erdős and Hajnal [9], [16], [19], states that if $\mathcal{G}$ contains a finite k-partite hypergraph, then

$$\mathfrak{a}^{+k} \to (\mathcal{G})^k_{\mathfrak{a}}. \tag{5}$$

For a nonempty family $\mathcal{G}$ of hypergraphs and for cardinals $\mathfrak{a}$ and $\mathfrak{b}$, the partition relation $\mathfrak{b} \to (\mathcal{G})^{<\omega}_{\mathfrak{a}}$ is said to hold if for every $\mathfrak{a}$-coloring of the finite subsets of a set of cardinality $\mathfrak{b}$, there is a monochromatic copy of a hypergraph $G \in \mathcal{G}$. In the trivial case that $\mathfrak{a}$ and $\mathfrak{b}$ are infinite cardinals and $\mathfrak{a} \geq \mathfrak{b}$, we have $\mathfrak{b} \to (\mathcal{G})^{<\omega}_{\mathfrak{a}}$ if and only if $\mathcal{G}$ contains a hypergraph with at most one edge. This can be seen by coloring each finite subset of the vertex set of cardinality $\mathfrak{b}$ a different color.

Theorem 4 *Let $\mathfrak{a}$ be an infinite cardinal and s a positive integer such that $\mathfrak{a}^{+s} \leq 2^{\mathfrak{a}}$. For a nonempty family $\mathcal{G}$ of finite hypergraphs that does not contain an s-hybrid hypergraph, we have*

$$\mathfrak{a}^{+s} \not\to (\mathcal{G})^{<\omega}_{\mathfrak{a}}.$$

Theorem 4 is proved in Section 3. We conjecture that Theorem 4 is in fact tight.

Conjecture 1 *Let $\mathfrak{a}$ be an infinite cardinal and s a positive integer such that $\mathfrak{a}^{+s} \leq 2^{\mathfrak{a}}$. For a nonempty family $\mathcal{G}$ of finite hypergraphs, we have*

$$\mathfrak{a}^{+s} \to (\mathcal{G})^{<\omega}_{\mathfrak{a}}$$

if and only if $\mathcal{G}$ contains an s-hybrid hypergraph.

We prove Conjecture 1 for the case $s = 1$ in Section 3.

We deduce Theorem 1 and Theorem 2 as corollaries of Theorem 4 and the Erdős-Hajnal partition relation (5). In fact, for a given system $\mathcal{L}$ of linear homogeneous equations and infinite cardinal $\mathfrak{a}$, we prove that there is an associated family $\mathcal{G}_L$ of finite hypergraphs such that $\mathcal{L}$ is $\mathfrak{a}$-regular over R if and only if $|R| \to (\mathcal{G}_{\mathcal{L}})^{<\omega}_{\mathfrak{a}}$. As we shall prove, $\mathcal{L}$ is separable if and only if $\mathcal{G}_{\mathcal{L}}$ contains a partite hypergraph. Moreover, if $\mathcal{L}$ is separable, then every hypergraph $G \in \mathcal{G}_{\mathcal{L}}$ is not an $(s(\mathcal{L}) - 1)$-hybrid, but there is an $s(\mathcal{L})$-partite hypergraph in $\mathcal{G}_{\mathcal{L}}$.

So far, we have implicitly assumed the axiom of choice. In Section 5, we demonstrate how heavily $\aleph_0$-regularity over $\mathbb{R}$ can depend on whether or not we assume the axiom of choice.

2 Definitions

We let $<$ denote the well-ordering of the ordinals. For a well-ordered set $\langle S, < \rangle$ and $a \in S$, define the *segment* $S_a = \{b \in S : b < a\}$. The *cardinality* $|S|$ of a set S is the least ordinal α such that there exists a bijection $\phi : S \to \alpha$. We let $<_S$ denote a well-ordering of S that is order-isomorphic to $|S|$, that is, there is a bijection $\phi : S \to |S|$ such that for all $x, y \in S$, we have $x <_S y$ if $\phi(x) < \phi(y)$. Notice that for the well-ordered set $\langle S, <_S \rangle$ and $a \in S$, the inequality $|S_a| < |S|$ is true. For a cardinal κ and set S, define

$$[S]^{\kappa} = \{T \subset S : |T| = \kappa\} \text{ and } [S]^{<\kappa} = \{T \subset S : |T| < \kappa\}.$$

For a set A, the *power set* $\mathcal{P}(A)$ is the set of subsets of A. For a well-ordered set $(A, <_A)$ and distinct subsets $X, Y \subset A$, let

$$\delta(X, Y) = \min\{a \in A : \text{exactly one of } X \text{ and } Y \text{ contains } a\}$$

and

$$X \prec Y \text{ if } \delta(X, Y) \in Y.$$

The linear ordering $\prec$ is called the *lexicographic ordering* of the power set $\mathcal{P}(A)$.

Let $\mathfrak{a}$ be an infinite cardinal, s a positive integer, A be a set of cardinality $\mathfrak{a}$, and $S \subset \mathcal{P}(A)$ be a set with cardinality $\mathfrak{a}^{+s}$ and well-ordering $>_S$. For a finite set $T \subset S$, we let t_0 be the largest element of T in the well-ordering $>_S$ of S. We let $S_1(T)$ denote the segment S_{t_0} of the well-ordered set $(S, >_S)$. For $1 \leq r < |T|$, we let t_r be the largest element of $T \cap S_r(T)$ in the well-ordering $<_{S_r(T)}$ of $S_r(T)$, and define the set $S_{r+1}(T)$ to be the segment

$S_r(T)_{t_r}$ of the well-ordered set $(S_r(T), >_{S_r(T)})$. Note that $t_i >_{S_i} t_j$ for $1 \leq i < j < |T|$ and $\mathfrak{b} > |S_1(T)| > \ldots > |S_{|T|-1}(T)|$. Define the linear ordering $>_1$ of the elements of T by $t_i >_1 t_j$ for $0 \leq i < j \leq |T|-1$. Notice that the set $S_s(T)$ is defined only by the largest s elements of T by ordering $>_1$. Define the finite set $c_2(T)$ of ordinals by

$$c_2(T) = \{\phi(t_i)\}_{s \leq i \leq |T|-1},$$

where ϕ is a bijection from $S_s(T)$ to the cardinal $|S_s(T)|$.

We define the coloring $c_3(T)$ to be the permutation $\pi : \{i\}_{i=0}^{|T|-1} \to \{i\}_{i=0}^{|T|-1}$ such that $t_{\pi(0)} \prec \ldots \prec t_{\pi(|T|-1)}$.

We define the coloring c_1 as follows

$$c_1(t) = (\delta_1, \delta_2, \ldots, \delta_{|T|-1}) \tag{6}$$

where we define $\delta_i := \delta(t_{\pi(i-1)}, t_{\pi(i)})$.

3 Hypergraph Partition Relations

Let $\mathfrak{a}$ be an infinite cardinal, s be a positive integer, and S be a set of cardinality $\mathfrak{a}^{+s}$. We give an $\mathfrak{a}$-coloring c of the elements of $[S]^{<\omega}$ such that the only monochromatic hypergraphs are s-hybrids. The coloring c is the product coloring $c = c_1 \times c_2 \times c_3$ of the three colorings defined in the previous section.

Theorem 5 *Assume $\aleph_0 \leq \mathfrak{a} < \mathfrak{b} \leq 2^{\mathfrak{a}}$ and S is a set of cardinality $\mathfrak{a}^{+s}$.*
(1) There exists an an $\mathfrak{a}$-coloring c_1 of $[S]^{<\omega}$ whose monochromatic subhypergraphs are partite.
(2) If $\mathfrak{b} = \mathfrak{a}^{+s}$, then there exists an $\mathfrak{a}$-coloring c of $[S]^{<\omega}$ whose monochromatic subhypergraphs are s-hybrids.

Proof:

(1) Every monochromatic hypergraph in the coloring c_1 is uniform since the color of each k-set is a $(k-1)$-tuple. Let $d_1 = (\delta_1, \ldots, \delta_{k-1})$ be one of the colors in the coloring c_1. Define the partition $S = \bigcup_{j=0}^{k-1} S_j$ by letting $v \in S_j$ if $\delta_i \in v$ for $i < j$ and $\delta_j \notin v$. From the definition of the coloring c_1, every edge in a monochromatic hypergraph of color d_1 has a vertex in each S_j for $1 \leq j \leq k$. Hence a monochromatic hypergraph in this $\mathfrak{a}$-coloring is partite.

(2) Let c be the product coloring $c = c_1 \times c_2 \times c_3$, where c_1, c_2, and c_3 are defined in the previous section. Since $\mathfrak{b} > |S_1(T)| > \ldots > |S_{|T|-1}(T)|$, then the set of colors of c_2 has cardinality at most $\mathfrak{a}$. The cardinality of the set of colors of c_3 is $\aleph_0$ and the cardinality of the set of colors of c_1 is $\mathfrak{a}$. Hence, the set of colors of c has cardinality $\mathfrak{a}$. By the colorings c_1 and c_3, for every color d of c, there is a positive integer k and partition $S = S_1 \cup \ldots \cup S_k$ of S such that every finite subset $T = \{t_i\}_{i=0}^{|T|-1}$ of S that is colored d by c has $|T| = k$ and $t_{\pi(i)} \in S_i$ for $1 \leq i \leq k$. By the colorings c_2, every s-tuple $(v_0, \ldots, v_{s-1})$ with $v_i \in S_{\pi^{-1}(i)}$ for $0 \leq i \leq s-1$

satisfies that there is at most one finite subset e of color d such that $\{v_0, \ldots, v_{s-1}\} \subset e$. Therefore, every monochromatic hypergraph in the coloring c is an s-hybrid. $\square$

For a hypergraph H and cardinal $\mathfrak{b}$, the hypergraph $\mathfrak{b}H$ consists of $\mathfrak{b}$ disjoint copies of H.

Lemma 1 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be infinite cardinals such that* $\mathfrak{a} < \mathfrak{b}$ *and* $\mathfrak{b}$ *is a regular cardinal. If* H *is a* k*-uniform hypergraph such that* $\mathfrak{b} \to (\{H\})^k_{\mathfrak{a}}$*, then* $\mathfrak{b} \to (\{\mathfrak{b}H\})^k_{\mathfrak{a}}$*.*

Proof: We have $\mathfrak{b} = \mathfrak{b} \times \mathfrak{b}$, so every $\mathfrak{a}$-coloring of the edges of the complete k-uniform hypergraph has at least $\mathfrak{b}$ monochromatic copies of H. Since $\mathfrak{b} > \mathfrak{a}$ and $\mathfrak{b}$ is regular, then there are $\mathfrak{b}$ monochromatic copies of H all of the same color. Hence, we have $\mathfrak{b} \to (\mathfrak{b}H)^k_{\mathfrak{a}}$. $\square$

For a hypergraph $H = (V, E)$ and vertex $v \in V$, define the neighborhood hypergraph $N(H, v) = (V', E')$ by

$$V' = \{w : w \in V/\{v\} \text{ and there is an edge } e \in E \text{ such that } \{v, w\} \subset e\}$$

and

$$E' = \{e/\{v\} : v \in e \in E\}.$$

We define the infinite k-uniform hypergraph $H(k, \mathfrak{a})$ recursively. The 1-uniform hypergraph $H(1, \mathfrak{a}) = (V, E)$ has $\mathfrak{a}^+$ vertices and $E = \{\{v\} : v \in V\}$. The k-uniform hypergraph $H(k, \mathfrak{a})$ consists of $\mathfrak{a}^+$ disjoint copies of the k-uniform graph which consists of a root vertex v whose neighborhood hypergraph is $H(k-1, \mathfrak{a})$.

Lemma 2 *Let* k *be a positive integer and* $\mathfrak{a}$ *an infinite cardinal. Every* $\mathfrak{a}$*-coloring of the finite subsets of a set of cardinality* $\mathfrak{a}^+$ *contains a monochromatic* $H(k, \mathfrak{a})$*.*

Proof: The proof is by induction on k. For $k = 1$, the result follows immediately from the transfinite pigeonhole principle. Consider a a positive integer k and assume the lemma is true for that k. Consider a vertex v of the $\mathfrak{a}^+$ vertices, and consider the edges of size $k+1$ containing v. By the induction hypothesis, there is a monochromatic hypergraph where every edge contains v and the neighborhood hypergraph of v is $H(k, \mathfrak{a})$. Combining this with Lemma 1, every $\mathfrak{a}$-coloring of the finite subsets of a set of cardinality $\mathfrak{a}^+$ contains a monochromatic $H(k+1, \mathfrak{a})$. By induction, for every positive integer k and $\mathfrak{a}$-coloring of the finite subsets of a set of cardinality $\mathfrak{a}^+$, there is a monochromatic $H(k, \mathfrak{a})$. $\square$

Every 1-hybrid k-uniform graph on at most $\mathfrak{a}^+$ vertices is a subhypergraph of $H(k, \mathfrak{a})$. Hence, Corollary 2 follows immediately from Lemma 2 and Theorem 5.

Corollary 2 *For every infinite cardinal* $\mathfrak{a}$ *and family* $\mathcal{G}$ *of hypergraphs, we have* $\mathfrak{a}^+ \to (\mathcal{G})^{<\omega}_{\mathfrak{a}}$ *if and only if* $\mathcal{G}$ *contains a* 1*-hybrid hypergraph on at most* $\mathfrak{a}^+$ *vertices.*

We next prove the Erdős and Hajnal polarized partition relation (5). A proof of this result can also be found in the papers of Schmerl [19] and Komjáth [16]. The complete k-partite hypergraph $P(k; n) = (V, E)$ is defined by $V = V_1 \cup \ldots \cup V_k$ with $|V_i| = n$ for $1 \leq i \leq k$, and $(v_1, \ldots, v_k) \in E$ if $v_i \in V_i$ for $1 \leq i \leq k$, and there are no other edges. Notice that every k-partite graph on n vertices is a subhypergraph of the hypergraph $P(k; n)$.

Lemma 3 *Let k and n be positive integers and $\mathfrak{a}$ an infinite cardinal. Every $\mathfrak{a}$-coloring of the edges of the complete k-uniform hypergraph on $\mathfrak{a}^{+k}$ vertices contains a monochromatic $P(k;n)$.*

Proof: The proof is by induction on k. For $k = 1$, this result follows immediately from the transfinite pigeonhole principle. Consider a a positive integer k and assume the lemma is true for that k. Consider an $\mathfrak{a}$-coloring of the complete $(k+1)$-uniform hypergraph $K^{(k+1)}_{\mathfrak{a}^{k+1}}$ on $\mathfrak{a}^{k+1}$ vertices. Partition the $\mathfrak{a}^{k+1}$ vertices into two sets, X and Y, such that $|X| = \mathfrak{a}^{k+1}$ and $|Y| = \mathfrak{a}^k$. For each $x \in X$, consider the edges of $K^{(k+1)}_{\mathfrak{a}^{k+1}}$ that include x and k vertices from Y. By the induction hypothesis, there is a monochromatic $(k+1)$-hypergraph such that each edge contains x and the neighborhood of x is a copy of $P(k;n)$ with vertices in Y. Make a pigeonhole for each copy of $P(k;n)$ with vertices in Y and each of the $\mathfrak{a}$ colors. There are $\mathfrak{a}^{+k}$ such pigeonholes. Place a vertex x in a pigeonhole if the neighborhood of x in the color of the pigeonhole contains the copy of $P(k;n)$ of the pigeonhole. Since there are $\mathfrak{a}^{k+1}$ such pigeons, then there are n vertices in one pigeonhole. Then these n vertices, along with the vertices of the copy of $P(k;n)$ of the pigeonhole, are the vertices of a monochromatic $P(k+1;n)$. By induction, we are done. □

4 Infinite color regularity of systems of linear homogeneous equations

Let V be a vector space over a field F. Consider a system of linear homogeneous equations

$$\mathcal{L} : \sum_{j=1}^{n} a_{ij} x_j = 0 \qquad \text{for } 1 \le i \le m. \tag{7}$$

with $a_{i,j} \in F$ for $1 \le i \le m$ and $1 \le j \le n$. For an infinite cardinal $\mathfrak{a}$, we call $\mathcal{L}$ *$\mathfrak{a}$-regular over V* if every $\mathfrak{a}$-coloring of V has a monochromatic solution to $\mathcal{L}$ in distinct variables.

Theorem 6 *Let $\mathfrak{a}$ be an infinite cardinal. If V is a vector space over a field F such that $|F| \le \mathfrak{a}$ and $|V| \le 2^{\mathfrak{a}}$, then $\mathcal{L}$ is $\mathfrak{a}$-regular over V if and only if $\mathcal{L}$ is separable and $|V| \ge \mathfrak{a}^{+s(\mathcal{L})}$. Moreover, there is an $\mathfrak{a}$-coloring of V such that there are no monochromatic solutions in distinct variables to all linear homogeneous systems of equations with coefficients in F that are not $\mathfrak{a}$-regular over V.*

Proof: If $|V| \le \mathfrak{a}$, then we can color every element of V a different color, and there will be no nontrivial systems of linear homogeneous equations with a monochromatic solution in distinct variables. Hence, we assume that $|V| > \mathfrak{a} \ge |F|$. In this case, we have that the dimension of V is equal to the cardinality of V.

A well known result, which follows from the axiom of choice, is that every vector space has a basis. Since V is a vector space, then there is a basis $B = \{b_i\}_{i \in I}$ for V, where I is a well-ordered set that is order isomorphic with the cardinal $|V|$.

Therefore, every nonzero element v of V has a unique basis representation

$$v = \sum_{d=1}^{k} f_d b_{j_d} \text{ with } f_d \in F/\{0\} \text{ and } j_1 < \ldots < j_k.$$

For $j \in I$, we define $\pi_j(v)$ to be the coefficient of b_j in the representation of v. For $1 \leq d \leq k$, define $b(v, d) := b_{j_d}$.

We view the finite sequence $w(v) = (f_1, \ldots, f_k)$ as a "weight" and the finite set $e(v) = \{b_{j_1}, \ldots, b_{j_k}\} \in [B]^{<\omega}$ as an edge of a weighted hypergraph with vertex set B. Thus, each element v can be thought of as a "weighted edge" of a weighted hypergraph. That is, we can associate each element $v \in V$ with the pair $(w(v), e(v))$. To each element $v \in V$, we assign the color $\mathcal{C}(v) = w(v) \times c(e(v))$, where $c(e(v))$ is the coloring of the elements of $[V]^{<\omega}$ already given in the previous section. Since the set $w(V)$ has cardinality $|F|$ and the set $c(e(V))$ has cardinality at most $\mathfrak{a}$, then the coloring $\mathcal{C}$ of V uses at most $\mathfrak{a}$ colors.

By the coloring w, if $\mathbf{x} = (x_1, \ldots, x_n)$ is a monochromatic solution to $\mathcal{L}$ in distinct variables, then we can write each x_p in its unique representation:

$$x_p = \sum_{d=1}^{k} f_d b_{j_{p,d}} \quad \text{with} \quad j_{p,1} < \ldots < j_{p,k}.$$

Also, for each coordinate $j \in I$ and each $i \in \{1, \ldots, m\}$, we have

$$\sum_{p=1}^{n} a_{i,p} \pi_j(x_p) = 0.$$

Define the hypergraph $H(\mathbf{x}) = (V(\mathbf{x}), E(\mathbf{x}))$ with vertex set $V(\mathbf{x}) \subset B$ defined by $b_j \in V(\mathbf{x})$ if $\pi_j(x_p) \neq 0$ for at least one p, $1 \leq p \leq n$ and edge set $E(\mathbf{x}) = \{e(x_1), \ldots, e(x_n)\}$.

By the coloring c_1, we can partition the vertex set $V(\mathbf{x}) = V_1 \cup \ldots \cup V_k$ such that each edge $e(x_p)$ contains exactly one vertex in each V_d for $1 \leq d \leq k$. For each d, let $v_{d,1} < v_{d,2} < \ldots < v_{d,|V_d|}$ denote the elements of V_d in order. Therefore, the set V_d of vertices corresponds to a balanced partition $\mathcal{P}_d = \cup_{i=1}^{|V_d|} P_i$ of the set $\{1, \ldots, n\}$, where $p \in P_i$ if $\pi_{v_{d,i}}(x_p) \neq 0$. Since $\mathbf{x} = (x_1, \ldots, x_n)$ is a monochromatic solution to $\mathcal{L}$ in distinct variables, then the edges $e(x_1)$, …, $e(x_n)$ are distinct subsets of V, and the corresponding collection $\{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$ of balanced partitions is separative. Hence $\mathcal{L}$ is separable. Moreover, for each subset $D \subset \{1, \ldots, k\}$ with $|D| < s(\mathcal{L})$, there is distinct p, $q \in \{1, \ldots, n\}$ such that $b(x_p, d) = b(x_q, d)$ for $d \in D$. But since $e(x_p)$ and $e(x_q)$ are distinct subsets for distinct p, $q \in \{1, \ldots, n\}$, then $H(\mathbf{x})$ can not be a d-hybrid with $d < s(\mathcal{L})$. Therefore, if $|V| < \mathfrak{a}^{+s(\mathcal{L})}$, then there are no monochromatic solutions to $\mathcal{L}$ in this $\mathfrak{a}$-coloring C of V.

Consider the subset $V' \subset V$ defined by

$$V' = \left\{ \sum_{b \in T} b : T \in [B]^{<\omega} \right\}.$$

Hence, we have $|V'| = |V|$. We can view a coloring of V' as a coloring of $[B]^{<\omega}$. If $\mathcal{L}$ is separable, then there is a separative collection $\{\mathcal{P}_1, \ldots, \mathcal{P}_{s(\mathcal{L})}\}$ of $s(\mathcal{L})$ balanced partitions. For $1 \le d \le s(\mathcal{L})$, write the balanced partition $\mathcal{P}_d$ of the set $\{1, \ldots, n\}$ as $\mathcal{P}_d = \cup_{i=1}^{|\mathcal{P}_d|} P_{d,i}$. Let $H = (V, E)$ denote the hypergraph with vertex set $V = \{P_{d,i} : 1 \le d \le s(\mathcal{L})$ and $1 \le i \le |\mathcal{P}_{d,i}|\}$ and edge set $E = \{e_p\}_{p=1}^{n}$ where $P_{d,i} \in e_p$ if and only if $p \in P_{d,i}$. From the definition, H is $s(\mathcal{L})$-partite with $s(\mathcal{L})$-partition $V = \mathcal{P}_1 \cup \ldots \mathcal{P}_{s(\mathcal{L})}$. Notice that every copy of the hypergraph $H \subset [B]^{<\omega}$ corresponds to a solution $\mathbf{x}$ to $\mathcal{L}$ in distinct variables with $H(\mathbf{x}) = H$. If $|V| \ge \mathfrak{a}^{s(\mathcal{L})}$, then every $\mathfrak{a}$-coloring of V has a monochromatic solution to $\mathcal{L}$ in distinct variables since $\mathfrak{a}^{+s(\mathcal{L})} \rightarrow \{H\}_{\mathfrak{a}}^{<\omega}$.

Therefore, $\mathcal{L}$ is $\mathfrak{a}$-regular over V if and only if $\mathcal{L}$ is separable and $|V| \ge \mathfrak{a}^{+s(\mathcal{L})}$. Moreover, the $\mathfrak{a}$-coloring C of V satisfies there are no monochromatic solutions in distinct variables to all linear homogeneous systems of equations with coefficients in F that are not $\mathfrak{a}$-regular over V. □

We now give a proof of Theorem 1.

Proof of Theorem 1: Consider a system $\mathcal{L}$ of linear homogeneous equations in a subring R of the complex numbers. We let $\mathcal{A}$ denote the set of coefficients $a_{i,j}$ of the system $\mathcal{L}$ of linear homogeneous equations. Let $\mathbb{Q}(\mathcal{A})$ denote the extension field that is generated by the elements of $\mathcal{A}$ and $\mathbb{Q}$. Since $\mathcal{A}$ is finite, the extension field $\mathbb{Q}(\mathcal{A})$ is a countable subfield of $\mathbb{C}$. Let R_1 be the ring generated by R and $\mathbb{Q}(\mathcal{A})$. We note that the cardinality of R_1 is the same as the cardinality of R. Since R_1 is a vector space over $\mathbb{Q}(\mathcal{A})$, then $\mathcal{L}$ is not $\mathfrak{a}$-regular over R_1 if $|R| < \mathfrak{a}^{+s(\mathcal{L})}$. Moreover, $\mathcal{L}$ is not $\mathfrak{a}$-regular over R if $|R| < \mathfrak{a}^{+s(\mathcal{L})}$.

Pick out a linearly independent set $B \subset R$ such that $|B| = |R|$. Consider the subset $R' \subset R$ defined by

$$R' = \left\{ \sum_{b \in T} b : T \in [B]^{<\omega} \right\}.$$

By the same argument as we gave in the proof of Theorem 6, if s is a positive integer such that $|R| \ge \mathfrak{a}^{+s}$, then every $\mathfrak{a}$-coloring of R' contains a monochromatic solution to every separable system $\mathcal{L}$ of linear homogeneous equations with separation number s. This finishes the proof of Theorem 1. □

5 Regularity without the axiom of choice

In 1942 Bernays [2] formulated the axiom known as the principle of dependent choice.

Definition: Principle of dependent choices If E is a binary relation on a nonempty set A and for every $a \in A$ there exists $b \in A$ with aEb, then there exists a sequence $a_1, a_2, \ldots, a_n, \ldots$ such that $a_n E a_{n+1}$ for every $n < \omega$.

The principle of dependent of dependent choice is usually denoted by DC. The axiom of choice implies DC, but not conversely (Theorem 8.2 in [15]). As usual, ZF is short for

Zermelo-Fraenkel system of axioms, and ZFC is short for Zermelo-Fraenkel system of axioms with the axiom of choice.

Definition: Axiom LM Every set of real numbers is Lebesgue measurable.

Axiom LM is not consistent with ZFC. However, In 1970, assuming the existence of an inaccessible cardinal, Solovay proved the following consistency result.

Theorem 7 (Solovay, [22]) *The system of axioms ZF + DC + LM is consistent.*

We call a system

$$\sum_{i=1}^{n} a_{ij}x_i = 0 \qquad \text{for } 1 \leq j \leq m$$

of linear homogeneous equations *homothetic* if $\sum_{i=1}^{n} a_{ij} = 0$ for $1 \leq j \leq m$. Rado [18] proved that if $\mathcal{L}$ is a system of linear homogeneous equations that is not homothetic, then there is a countable coloring of the real numbers without a monochromatic solution to $\mathcal{L}$ in distinct variables. The following theorem classifies those systems of linear homogeneous equations that are $\aleph_0$-regular in ZF+DC+LM.

Theorem 8 *In ZF+DC+LM, a system $\mathcal{L}$ of homogeneous linear equations is $\aleph_0$-regular over $\mathbb{R}$ if and only if $\mathcal{L}$ is homothetic and there is a solution to $\mathcal{L}$ in distinct variables.*

This classification is considerably different from the classification in ZFC given by Theorem 1. For example, $x_1 + x_2 = 2x_3$ is $\aleph_0$-regular over $\mathbb{R}$ in ZF+DC+LM but not $\aleph_0$-regular over $\mathbb{R}$ in ZFC. The proof of Theorem 8 follows from a result of Ceder [4].

If $S \subset \mathbb{R}^n$, then a *homothetic copy* of S is a set $aS + b = \{as + b : s \in S\}$ where $a, b \in \mathbb{R}$ and $a \neq 0$. Notice that a system $\mathcal{L}$ of linear homogeneous equations is homothetic if and only if for every solution $(x_1, \ldots, x_n)$ of $\mathcal{L}$, we have $(ax_1 + b, \ldots, ax_n + b)$ is also a solution of $\mathcal{L}$ for all a and b in $\mathbb{R}$. Hence the solution space of a system $\mathcal{L}$ of linear equations is closed under taking homothetic copies if and only $\mathcal{L}$ is homothetic. The major part of the proof of Theorem 8 is the following theorem of Ceder [4].

Theorem 9 (Ceder 1969) *If S is a finite subset of $\mathbb{R}^n$, then every countable coloring of $\mathbb{R}^n$ with each color class Lebesgue measurable contains a monochromatic homothetic copy of S.*

Since the set of solutions to a linear homogeneous system of equations is closed by homothetic copies, then Theorem 8 follows from Theorem 9.

Now we consider finite colorings of the real numbers. Fix q a rational number other than -1, 0, or 1. Radoičić and the author [13] proved that in ZF+DC+LM, every 3-coloring of the nonzero real numbers contains a monochromatic solution to $x_1 + qx_2 = q^2x_3$, but in ZFC, there is a 3-coloring of the nonzero real numbers without a monochromatic solution to $x_1 + qx_2 = q^2x_3$. Radoičić and the author also proved that in ZF+DC+LM, every 4-coloring of the nonzero real numbers contains a monochromatic solution to $x_1 + 2x_2 + 4x_3 = x_4$, but in ZFC, there is a 4-coloring of the nonzero real numbers without a monochromatic solution to $x_1 + 2x_2 + 4x_3 = x_4$.

To summarize, the results mentioned in this section demonstrate how regularity can depend on whether or not we assume the axiom of choice.

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