1.1.25

Proof. Let $C^{(j)}$ be the jth column of matrix $A$, then

$$A = [C^{(1)} \cdots C^{(n)}]. \quad (1)$$

Now, consider the matrix multiplication in the following form

$$b = Ax = x_1 C^{(1)} + \cdots + x_n C^{(n)}, \quad (2)$$

which is a linear combination of $A$. \hfill \Box

1.2.4

Proof. Use contradiction, assume there is a nonzero $y$ such that $Ay = 0$. Multiply $A^{-1}$ on both sides, one has $y = A^{-1} \cdot 0 = 0$ which contradicts with $y \neq 0$. \hfill \Box

1.3.4

• Step 1
  
  $y_1 = 1.$

• Step 2
  
  $y_2 = \frac{3 - (-2y_1)}{2} = 2.$

• Step 3
  
  $y_3 = \frac{2 - 3y_1 - y_2}{y_1} = 3.$

• Step 4
  
  $y_4 = \frac{9 - 4y_1 - y_2 - (-3y_3)}{3} = 4.$

Thus, $Y = [1 \ 2 \ 3 \ 4]^T.$
1.4.15

Proof. (a). For any $x \in \mathbb{R}^2$ and $x \neq 0$, consider $x^T A x = 4x_1^2 + 9x_2^2 > 0$, thus $A$ is definite positive.

(b). Apply the method on the book, $R = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(c). For those diag elements of $R$, we do not force them to be positive square root, then the other three upper triangular matrices can be derived

$$R_2 = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, \quad R_3 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$  \hspace{1cm} (3)

(d). Since for each $r_{ii}$, once it is determined, the $i$th row of $R$ is uniquely determined. Thus, for an $n$ by $n$ matrix, there are two options for each row of matrix factor $R$, so $2^n$ possibilities totally.

1.5.9

Proof. Assume the width $s = k$ where $k << n$. Then, firstly consider the first $k$ steps for solving from $x_n$ to $x_{n-k+1}$, we have

$$x_i = (y_i - \sum_{j=i+1}^{n} r_{i,j}x_j)/r_{i,i}, \quad n - k + 1 \leq i \leq n.$$  \hspace{1cm} (4)

Thus, the flops of first $k$ steps is $f_1 = 1 + 3 + \cdots + (2k - 1) = k^2$.

Next, consider the rest $n - k$ steps for solving from $x_{n-k}$ to $x_1$, since all $r_{i,j} = 0$ when $j - i > k$, we have

$$x_i = (y_i - \sum_{j=i+1}^{i+k} r_{i,j}x_j)/r_{i,i}, \quad 1 \leq i \leq n - k.$$  \hspace{1cm} (5)

Thus, the flops of the rest steps is $f_2 = (n - k) \cdot (2k + 1)$.

Finally, the total flops is $f(n) = f_1 + f_2 = (n - k)(2k + 1) + k^2$. Consider $k << n$, $k^2 \approx 0$, and therefore $f(n) \approx (n - k)(2k + 1) \approx 2nk$.

1.7.10

(a). Use matlab command `det()` to find the determinants of four submatrice.
(b). Apply the Gaussian elimination for matrix $A$, the result upper triangular matrix

$$U = \begin{bmatrix}
-6 & -1 & 2 & -3 \\
0 & 2/3 & -1/3 & 2 \\
0 & 0 & -1/2 & 3 \\
0 & 0 & 0 & -3 \\
\end{bmatrix}.$$  

\hspace{1cm} (6)

and

$$y = \begin{bmatrix}
-14 \\
8(1/3) \\
10(1/2) \\
-12 \\
\end{bmatrix}.$$  

\hspace{1cm} (7)

(c). Refer to 1.3.4, the solution is $x = [1 \ 2 \ 3 \ 4]^T$. 