HW2 Solution of Math170A

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2.1.32

Proof.

(i) $\|x\|_\infty \leq \|x\|_2$.
Since $\|x\|_\infty = |x_j| = \sqrt{x_j^2}$, where x_j is the largest element of x. Then, $\|x\|_\infty \leq \sqrt{x_1^2 + \cdots + x_n^2} = \|x\|_2$.

(ii) $\|x\|_2 \leq \|x\|_1$.
Take square on both sides, we have $\|x\|_2^2 = x_1^2 + \cdots + x_n^2 \leq (|x_1| + \cdots + |x_n|)^2 = \|x\|_1^2$.

(iii) $\|x\|_1 \leq \sqrt{n}\|x\|_2$.
Rewrite 1-norm as following,
$\|x\|_1 = |x_1| + \cdots + |x_n| = \sum_{i=1}^n |x_i| \cdot 1$.
By Theorem 2.1.6 Cauchy-Schwarz inequality, $\|x\|_1 \leq (\sum_{i=1}^n x_i^2)^{1/2}(\sum_{j=1}^n 1^2)^{1/2} = \sqrt{n}\|x\|_2$.

(iv) $\sqrt{n}\|x\|_2 \leq n\|x\|_\infty$.
$\sqrt{n}\|x\|_2 = \sqrt{n}\sqrt{\sum_{i=1}^n |x_i|^2} \leq \sqrt{n}\sqrt{\sum_{i=1}^n \|x\|_\infty^2} \leq n\|x\|_\infty$.

□

2.2.6

Proof.

(a).
Since ${A^{-1}}^{-1} = A$, we have $\kappa(A^{-1}) = \|A^{-1}\|\|{A^{-1}}^{-1}\| = \|A\|\|A^{-1}\| = \kappa(A)$

(b).
Since $(cA)^{-1} = \frac{1}{c}A^{-1}$, one has $\kappa(cA) = \|cA\|\|(cA)^{-1}\| = \|cA\|\|\frac{1}{c}A^{-1}\| = |c||1/c|\|A\|\|A^{-1}\| = \kappa(A)$.

□

2.2.24

(a). To calculate A^{-1}, solve $Ax = [10]^T$ and $x = [375\, -376]^T$ is the 1st column of A^{-1}, then solve $Ay = [01]^T$ and $y = [-187\; 187.5]^T$ is the 2nd column of A^{-1}. Thus,

$$A^{-1} = \begin{bmatrix} 375 & -187 \\ -376 & 187.5 \end{bmatrix}.$$

For $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$, recall the infinity-norm of matrix is given as $\|A\| = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$, which is simply the maximum absolute row sum of the matrix. In our case here, $\|A\|_\infty = 1502$ and $\|A^{-1}\|_\infty = 563.5$, thus $\kappa_\infty(A) = 8.4638 \cdot 10^5$ which indicates this matrix is a bad matrix.

(b). Let $x = [1\; -1]^T$ and $\delta b = [0.1\; 0.1]^T$, thus $b = Ax = [1\; 2]^T$ and $\delta x = A^{-1}\delta b = [18.8\; -18.85]^T$.

(c). Let $x = [1\; -1]^T$ and $\delta x = [0.1\; 0.1]^T$, thus $b = Ax = [1\; 2]^T$ and $\delta b = A\delta x = [75.9\; 152.2]^T$.

3.1.5

'(a). The system is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1.5 \\ 1 & 2 \\ 1 & 2.5 \\ 1 & 3 \end{bmatrix}; \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad b = \begin{bmatrix} 1.1 \\ 1.2 \\ 1.3 \\ 1.3 \\ 1.4 \end{bmatrix}.$$

(b). Follow the problem.

(c). Type the following commands,

```
plot(A(:,2),y,'o')
hold on
plot(A(:,2),x(1)*A(:,1)+x(2)*A(:,2),'-')
```

.

(d). $r = b - Ax$, and use command **norm(r,2)** to compute 2-norm of r.

3.2.8

Proof. Review that for orthogonal matrix, $Q^{-1} = Q^T$, and the p-norm of matrix is $\|Q\|_p = \max_{\forall x\neq 0} \frac{\|Qx\|_p}{\|x\|_p}$. Thus, $\|Q\|_2 = \max \frac{\|Qx\|_2}{\|x\|_2} = \max \frac{\sqrt{x^TQ^TQx}}{x^Tx} = 1$. Similarly, $\|Q^{-1}\|_2 = 1$, and $\kappa_2(Q) = \|Q\|_2\|Q^{-1}\|_2 = 1$. □

3.2.14

First, find rotator Q^T. Let $\cos\theta = \frac{a_{11}}{\sqrt{a_{11}^2+a_{21}^2}} = 2/\sqrt{29}$ and $\sin\theta = \frac{a_{21}}{\sqrt{a_{11}^2+a_{21}^2}} = 5/\sqrt{29}$, then

$$Q^T = \begin{bmatrix} 2/\sqrt{29} & 5/\sqrt{29} \\ -5/\sqrt{29} & 2/\sqrt{29} \end{bmatrix}.$$

Then, consider the system $Rx = Q^T b$, where

$$R = Q^T A = \begin{bmatrix} 5.3852 & 7.6135 \\ 0 & -0.1857 \end{bmatrix}; \quad Q^T b = \begin{bmatrix} 31.3825 \\ -0.3714 \end{bmatrix}.$$

Finally, use backward substitution to solve $Rx = \hat{b}$ and the solution is

$$x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

3.3.7

(a). Since the least square solution of overdetermined system is to minimize the sum of distance to each data points, here the middle point 7 is a reasonable guess.

(b).

Similarly as perious problem, find rotator, let $\cos\theta = \frac{a_{11}}{\sqrt{a_{11}^2+a_{21}^2}} = 1/\sqrt{2}$ and $\sin\theta = \frac{a_{21}}{\sqrt{a_{11}^2+a_{21}^2}} = 1/\sqrt{2}$, then

$$Q^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Then, consider the new system $Rx = Q^T b$, where $R = Q^T A = [\sqrt{2}\ 0]^T$ and $Q^T b = [14/\sqrt{2}\ 4/\sqrt{2}]^T$. For overdetermined system, we only consider the 1st nth equations of the system, in this case the 1st equation

$$\begin{bmatrix} \widehat{R} \\ 0 \end{bmatrix} [x] = \begin{bmatrix} \widehat{Q^T b} \\ 4/\sqrt{2} \end{bmatrix}.$$

i.e solve $\sqrt{2}x = 14/\sqrt{2}$ and the least square solution is $x = 7$.

3.4.22

(a)

Step one: $\widehat{q_1} = v_1$.

Step two: $q_1 = v_1/\|v_1\|_2 = v_1/6 = [1/2\ -1/2\ 1/2\ -1/2]^T$.

Step three: $\widehat{q_2} = v_2 - \langle v_2, q_1 \rangle q_1 = v_2 + q_1 = [3/2\ \ 3/2\ \ 7/2\ \ 7/2]^T$, where $r_{12} = \langle v_2, q_1 \rangle = -1$.

Step four: $q_2 = \widehat{q_2}/\|\widehat{q_2}\| = [3/2\sqrt{29}\ \ 3/2\sqrt{29}\ \ 7/2\sqrt{29}\ \ 7/2\sqrt{29}]$.

(b)

Now define $Q = [q_1\ q_2]$, because of $V = QR$, thus

$$R = Q^T V = \begin{bmatrix} 6 & -1 \\ 0 & \sqrt{29} \end{bmatrix}.$$

3.5.23

Instead of using QR decompsition to find least square solution, we can solve normal equations, which is

$$A^T A x = A^T b.$$

In this problem, $A^T A = 2$ and $A^T b = 14$, thus $x = 7$.