HW3 Solution of Math170A

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Feb 15th, 2012

4.1.6

Proof.

(i) It is not hard to see

$$AV = \begin{bmatrix} Av_1 & Av_2 & \cdots & Av_m \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 & \cdots & \sigma_r u_r & 0 & \cdots & 0 \end{bmatrix} = U\Sigma,$$

which is exactly the left half of equations (4.1.4), where u_i and v_i are ith column of U and V respectively. Similarly, from the right half of (4.1.4) we can observe $A^T U = V\Sigma^T$.

(ii) Since both U and V are orthogonal matrices, we have $A = AVV^T = U\Sigma V^T$ and $A = UU^T A = U\Sigma V^T$, i.e they imply the SVD of A.

□

4.2.12

(a)

Since rnak(A)=m and the SVD of A is given,

$$(A^T A)^{-1} = (V\Sigma^T U^T U\Sigma V^T)^{-1} = (V\Sigma^T\Sigma V^T)^{-1} = V(\Sigma^T\Sigma)^{-1}V^T = V\widehat{\Sigma}_1 V^T,$$

where

$$\widehat{\Sigma}_1 = \begin{bmatrix} 1/\sigma_1^2 & & & \\ & 1/\sigma_2^2 & & \\ & & \ddots & \\ & & & 1/\sigma_m^2 \end{bmatrix}_{m\times m}.$$

$$(A^TA)^{-1}A^T = V\widehat{\Sigma}_1 V^T V \Sigma^T U^T = V\widehat{\Sigma}_1 \Sigma^T U^T = V\widehat{\Sigma}_2 U^T,$$

where

$$\widehat{\Sigma}_2 = \begin{bmatrix} 1/\sigma_1 & & & & \vdots & \cdots \\ & 1/\sigma_2 & & & 0 & \cdots \\ & & \ddots & & \vdots & \cdots \\ & & & 1/\sigma_m & \vdots & \cdots \end{bmatrix}_{m\times n}.$$

$$A(A^TA)^{-1} = U\Sigma V^T V\widehat{\Sigma}_1 V^T = U\Sigma\widehat{\Sigma}_1 V^T = U\widehat{\Sigma}_3 V^T,$$

where

$$\widehat{\Sigma}_3 = \begin{bmatrix} 1/\sigma_1 & & & \\ & 1/\sigma_2 & & \\ & & \ddots & \\ & & & 1/\sigma_m \\ \cdots & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}_{n\times m}.$$

$$A(A^TA)^{-1}A^T = U\widehat{\Sigma}_3 V^T V \Sigma^T U^T = U\widehat{\Sigma}_3 \Sigma^T U^T = U\widehat{\Sigma}_4 U^T,$$

where

$$\widehat{\Sigma}_4 = \begin{bmatrix} I_{m\times m} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}.$$

(b)

Apply Theorem 4.2.1 which says 2-norm of a matrix is the greatest singular value, we can easily to find the greatest singlular of each of them, i.e the lagerest diag element of $\widehat{\Sigma}_1$, $\widehat{\Sigma}_2$, $\widehat{\Sigma}_3$ and $\widehat{\Sigma}_4$, hence $\|(A^TA)^{-1}\|_2 = 1/\sigma_m^2$, $\|(A^TA)^{-1}A^T\|_2 = 1/\sigma_m$, $\|A(A^TA)^{-1}\|_2 = 1/\sigma_m$ and $\|A(A^TA)^{-1}A^T\|_2 = 1$.

4.3.8

Proof.

By the form given in equation 4.3.5,

$$A^\dagger = V\Sigma^\dagger U^T = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \begin{bmatrix} 1/\sigma_1 & & & & \\ & \ddots & & & \\ & & 1/\sigma_r & & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix} = \sum_{i=1}^{r} \sigma_i^{-1} v_i u_i^T.$$

Remark, the way to test pseudoinverse I mentioned on my section that testing $A^\dagger A = I$ is **wrong**, to find the pseudoinverse just follow the form in equation 4.3.5. □

4.3.9

(a)

Here it is allow to use matlab to find SVD of A according to problem 4.1.15. However since the rank of A is one, it is not hard to see the condensed form of A is

$$A = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \begin{bmatrix} \sqrt{70} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix},$$

based on the method mentioned in equation 4.1.8 of page 262 that $span\{v_1\} = \mathcal{R}(A^T)$ and $span\{u_1\} = \mathcal{R}(A)$. Remark, the columns of orthogonal matrix are orthonormal to each other, so after you find the basis of certain space, remember to normalize them.

(b)

By equation 4.3.6,

$$A^\dagger = \widehat{V}\widehat{\Sigma}^{-1}\widehat{U}^T = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{70} \end{bmatrix} \begin{bmatrix} 1/\sqrt{14} & 2/\sqrt{14} & 3/\sqrt{14} \end{bmatrix} = \frac{1}{70} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}.$$

(c)

To find least square solution, if A is full rank, QR decomposition, normal sysytem and SVD are able to employed. Here, since rank(A)=1 which is not full rank, thus only SVD approach can be applied, by Theorem 4.3.7, the least square solution is given by $x = A^\dagger b = [6/70 \;\; 12/70]$.

(d)

Review the kernal of A is $\mathcal{N}(A) = \{x \in \mathbb{R}^m \mid Ax = 0\}$.Now, consider

$$Ax = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 6x_2 \end{bmatrix} = 0.$$

From it, we know if $x = c(-2\ \ 1)^T$, then it is a solution of $Ax = 0$, thus the basis of $\mathcal{N}(A)$ is $(-2\ \ 1)^T$.

(e)

To find all least square solutions, we consider the minimization problem $min\|b - Ax\|_2$.

$$f(x) \equiv \|b - Ax\|_2^2 = (1 - x_1 - 2x_2)^2 + (1 - 2x_1 - 4x_2)^2 + (1 - 3x_1 - 6x_2)^2.$$

Let $\frac{\partial f}{\partial x_1} = 0$ and $\frac{\partial f}{\partial x_2} = 0$, one has $7x_1 + 14x_2 = 3$, i.e for any $c \in \mathbb{R}$, $(c\ \frac{3-7c}{14})^T$ will be a least square solution of this system.

5.1.18

In order to solve this problem, it is suggested to read from bottom of page 298 to page 299 just above this problem, all concepts needed are introduced there.

(i) Rewrite this 2nd order ODE to 1st order ODE system.

Define $x_1 = \theta$ and $x_2 = \theta'$, then one has

$$\begin{cases} x_1' = x_2 \\ x_2' = -k_2 \sin x_1 - k_1 x_2. \end{cases}$$

In the vector form, it can be denoted as

$$\dot{x} = F(x).$$

(ii) Find equilibrium points.

To find equilibrium points, just set $F(x) = 0$, i.e solve

$$\begin{cases} x_2 = 0 \\ -k_2 \sin x_1 - k_1 x_2 = 0. \end{cases}$$

Then, we have $x_1 = n\pi$ and $x_2 = 0$, obvious they are corresponding to $\theta = n\pi$ and $\theta' = 0$.

(iii) Show when n is even, $n\pi$ is asymptotically stable, otherwise unstable.

To show it, find jacobian matrix of F,

$$J(F) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_2 \cos n\pi & -k_1 \end{bmatrix}.$$

Then find eigenvalues of $J(F)$ by solving $\det(J(F) - \lambda I) = 0$, and the characteristic polynomial is

$$\lambda^2 + k_1\lambda + k_2 \cos n\pi = 0.$$

Since k_1 and k_2 are positive, thus to make sure both two eigenvalues are negative (the sufficient condition of asymptotically stable), one need to show the larger root of characteristic poly is negative which is

$$\frac{-k_1 + \sqrt{k_1^2 - 4k_2 \cos n\pi}}{2} < 0.$$

Thus, only if $\sqrt{k_1^2 - 4k_2 \cos n\pi} < k_1$, take square on both sides, one has $\cos n\pi > 0$ which implies n is even.

(iv) Explain physically.

Imagine the original position of pendulum is vertical down to the earth (the case that $\theta = n\pi$ when n is even), if you perturb it a little away from original place, then as time goes to infinity, the pendulum will stop back to the original place by gravity and losing of energy. However if the original position is horizontal to the ground (n is odd), after small perturbation, the pendulum won't go back to the original place.

5.1.19

(a)

It is not hard to see

$$x(t) = \sum_{i=1}^{n} c_n e^{\lambda_i t} v_i = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = V e^{\Lambda t} c.$$

Noitce that $Ve^{\Lambda t}c$ is the general solution of homogeneous system given at equation 5.1.8.

(b)

To show the given solution is a solution of the nonhomogeneous system, the most direct way is plug the given solution back into the system, then we have

$$\begin{aligned} Ax - b &= A(z + Ve^{\Lambda t}c) - b \\ &= Az + AVe^{\Lambda t}c - b \quad \text{Notice } \mathrm{Ve}^{\Lambda t}\mathrm{c} \text{ is the general solution of homogeneous system} \\ &= b + (Ve^{\Lambda t})' - b \\ &= (z + Ve^{\Lambda t})' = \dot{x}. \end{aligned}$$

Thus, the given solution is a solution of nonhomogeneous system.

(c)

Set $t = 0$ for the general solution from **(b)**, we have

$$\hat{x} = x(0) = z + VIc = z + Vc.$$

5.1.23

It is suggested to read page 297 and Example 1.2.10 before doing this probem.

(a)

Since all the mass of carts are 1, thus Newton's second law is $a = F$. Now look at the left cart, the acceleration of it is x_1'' and the velocity of it is x_1'. Then the net force of it including four forces which are pull from left spring $-4x_1$, the pull from right spring $4(x_2 - x_1)$, the artificial force 1 and the friction $-k_1x_1'$, totally, we have $x_1'' = -4x_1 + 4(x_2 - x_1) - k_1x_1' + 1$. Similary the ODE of the other two carts can be obtained,

$$\begin{aligned} x_1'' &= -k_1x_1' - 8x_1 + 4x_2 + 1 \\ x_2'' &= -k_2x_2' + 4x_1 - 8x_2 + 4x_3 + 2 \\ x_3'' &= -k_3x_3' + 4x_2 - 8x_3 + 3 \end{aligned}$$

(b)

Let $x_4 = x_1'$, $x_5 = x_2'$ and $x_6 = x_3'$, we can rewrite the system into a 1st order ODE system

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -8 & 4 & 0 & -k_1 & 0 & 0 \\ 4 & -8 & 4 & 0 & -k_2 & 0 \\ 0 & 4 & -8 & 0 & 0 & -k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -2 \\ -3 \end{pmatrix}.$$

(c)

Solve $Ax - b = 0$, we have the solution

$$x = \begin{bmatrix} 5/8 \\ 1 \\ 7/8 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(d)

Since the system is initially at rest, the initial condition is $x(0) = 0$. To apply the general solution, we should find the eigenvalues and eigenvectors of matrix A. For this problem, it is too hard to find eigenvalues from a 6×6 matrix, hence I suggest use matlab, here is my code

```
[V D]=eig(A)
z=A\b
c=V\ (-z)
for i=1:20
DD=D;
for j=1:6
DD(j,j)=exp(D(j,j)*i);
end
x(:,i)=z+V*DD*c;
end
for i=1:20
t(i)=i;
end
plot(t,x(1,:))
plot(t,x(2,:))
plot(t,x(3,:))
```

5.2.6

(a)

The characteristic poly is from $\det(B - \lambda I) = 0$, thus here

$$\det \begin{bmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} = (\lambda - 1)(\lambda - 2) = 0.$$

Thus, we have two distinct eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$.

(b)

Similarly we have the characteristic poly is $(\lambda - 1)^2 = 0$, thus C has two equally eigenvalues which is 1.

5.2.17

Frist, recall that

$$\Sigma^T\Sigma = \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}_{m\times m}$$

$$\Sigma\Sigma^T = \begin{bmatrix} \hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$$

where

$$\begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_r^2 \end{bmatrix}_{r\times r.....} .$$

Thus, we have $A^TA = V\Sigma^TU^TU\Sigma V^T = V\Sigma^T\Sigma V^T$, multiply V from right on both sides, we have

$$A^TA \begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix} = V\Sigma^T\Sigma = \begin{bmatrix} \sigma_1^2 v_1 & \cdots & \sigma_r^2 v_r & 0 & \cdots & 0 \end{bmatrix}.$$

Similarly, we have

$$AA^T \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} = U\Sigma^T\Sigma = \begin{bmatrix} \sigma_1^2 u_1 & \cdots & \sigma_r^2 u_r & 0 & \cdots & 0 \end{bmatrix}.$$

Remark, this problem actually give us a method to find svd by hand!