HW4 Solution of Math170A

Shi(Fox) Cheng

Feb 27th, 2012

5.3.6

(i) Find eigenvalues.
Consider

$$\det(A - \lambda I) \;=\; \lambda^2 - 11\lambda + 17 \;=\; 0$$

Apply quadratic formula one can easily find two roots $\lambda_1 = 9.1401$ and $\lambda_2 = 1.8599$.

(ii) Find dominant eigenvector.
Dominant eigenvector is the eigenvector w.r.s.p to λ_1. Look at $(A - \lambda_1)v = 0$, one has $v = c(1 \;\; 0.1401)^T$, WLOG pick $c = 1$ which gives us the same result of example 5.3.5.

(iii) Calculate convergent ratio $\|q_{j+1} - v\|/\|q_j - v\|$.
Skipped.

(iv) Why $\|q_{j+1} - v\|/\|q_j j - v\| = |\lambda_2/\lambda_1|$.
Since for two by two matrix, it has only two eigenvalues. Refer to the prove of the error convergent ratio estimator on the bottom of page 314, it is exactly $|\lambda_2/\lambda_1|$.

(v) Why $\|q_{j+1} - v\|/\|q_j - v\| \neq |\lambda_2/\lambda_1|$ if matrix is larger.
The reason is that refer to the same prove on page 314, for larger matrix, the ratio $|\lambda_2/\lambda_1|$ becomes the upper bound of exact convergent ratio.

5.3.17

(a)

The process of Shift-and-Invert Strategy for power method, i.e compute q_{j+1} from q_j, one need do

(i): $\widehat{q}_{j+1} = (A - \rho I)^{-1} q_j$.
(ii): Pick $s_{j+1} = \|\widehat{q}_{j+1}\|_\infty$, i.e the maximum abs element of $\widehat{q}_{j+1}$.
(iii): $q_{j+1} = \widehat{q}_{j+1}/s_{j+1}$.
Proceed this process from q_0 for a few steps.

(b)

To calculate the eigenvalues and dominant eigenvector of A, do the exactly same calculation as problem 5.3.6, and compare the convergence ratio $\|q_{j+1} - v\|/\|q_j - v\|$ with theoretical convergence ratio $|(\lambda_1 - 8)/(\lambda_2 - 8)|$.

5.4.5

Let $D = diag(d_1, d_2, \cdots, d_n)$ and $v_i = (0, 0, \cdots, 1, 0 \cdots)^T$ where the ith element is one. Since

$$Dv_i = d_i v_i, \quad \forall i = 1, 2, \cdots, n.$$

It implies each diag element d_i is an eigenvalue with the eigenvector v_i. Clearly, those v_is are linear independent for $i = 1$ to n, thus n linear independent eigenvalues are found.

5.4.25

Proof.

Since similarity transformation preserves the eigenvalues, which means for $B = S^{-1}AS$, if λ_i is an eigenvalue of A, then it is one of B as well. Thus, this problem is just to show the Dim of eigenspace w.r.s.p to λ_i of A(Denote this eigenspace as E_{iA}) equals to the Dim of eigenspace w.r.s.p to λ_i of B(Denote this eigenspace as E_{iB}). Assume $\dim(E_{iA}) = d$, i.e any eigenvector w.r.s.p to λ_i of A, say v has $v \in span\{w_1, \cdots, w_d\}$, where $\{w_1, \cdots, w_d\}$ is the basis of E_{iA}, and can be written in linear combination form $v = \sum_{j=1}^{d} c_j w_j$.

Now show the set $\{S^{-1}w_j\}_{j=1,\cdots,d}$ is a linear independent set.
Consider

$$\sum c_j S^{-1} w_j = S^{-1} \sum c_j w_j = 0,$$

since S^{-1} is nonsingular, thus the above equation equals to zero is equivalent to $\sum c_j w_j = 0$, by the linear independency of w_js, one has the coefficents c_js are 0. Hence set $\{S^{-1}w_j\}_{j=1,\cdots,d}$ is a linear independent set.

Claim $E_{iB} = span\{S^{-1}w_j\}_{j=1,\cdots,d}$.
(i) Show $E_{iB} \subset span\{S^{-1}w_j\}_{j=1,\cdots,d}$.
Pick any element $v^\star \in E_{iB}$, one has $Bv^\star = \lambda_i v^\star$. Since $SB = AS$, then $SBv^\star = \lambda_i(Sv^\star) = A(Sv^\star)$, i.e

$Sv^\star \in E_{iA}$, thus it can be written as $Sv^\star = \sum c_j w_j$, and obviously $v^\star = S^{-1}\sum c_j w_j = \sum c_j(S^{-1}w_J) \in span\{w_j\}_{j=1,\cdots,d}$.
(ii) Show $E_{iB} \supset span\{S^{-1}w_j\}_{j=1,\cdots,d}$.
For any $v^\star = \sum c_j(S^{-1}w_j)$, we have $Bv^\star = BS^{-1}\sum c_j w_j = S^{-1}A\sum c_j w_j = \lambda_i S^{-1}\sum c_j w_j = \lambda_i v^\star$, thus it is in E_{iB}.
All for these two points, we show our claim, and it implies $\dim(E_{iB}) = d$.

□

5.4.26

If A is semisimple matrix, then its eigenvalue has geometric mutiplicity equals to its algebraic multiplicity.

Proof.

Assume the algebraic multiplicity of λ_i is r, i.e there are r eigenvalues equals to λ_i. We denote the eigenspace w.r.s.p to λ_i as E_i that the geometric mutiplicity of $\lambda_i = \dim(E_i)$
(i) Show $\dim(E_i) \geq r$.
By theorem (5.4.6), one can decomposite matrix as

$$A = V^{-1}\begin{bmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_i & & & & \\ & & & \ddots & & & \\ & & & & \lambda_i & & \\ & & & & & \lambda_{i+1} & \\ & & & & & & \ddots \end{bmatrix}[v_1 \cdots v_n],$$

where the multiple eigenvalue λ_i appears r times on the diag of D and all the column of V from v_i to v_{i+r-1} are linear independent eigenvectors w.r.s.p to λ_i, thus there are at least r linear independent vectors in E_i, i.e $\dim(E_i) \geq r$.

(ii) Show $\dim(E_i) \leq r$.
Assume $\dim(E_i) = r + d > r$, and the basis of E_i is $\{w_j\}_{j=1,\cdots,r+d}$. Recall that those columns of V except from v_i to v_{i+r-1} are eigenvectors corresponding to other eigenvalues $\lambda_j \neq \lambda_i$, for simplicity, reindex them as $\{v_1, \cdots, v_{n-r}\}$. Now show the set $\{w_1, \cdots, w_{r+d}, v_1, \cdots, v_{n-r}\}$ is a linear independent set. Consider $\sum c_j w_j + \sum l_k v_k = 0$, use contradiction and let this set be linear dependent, then one has a nonzero solution for those coefficents and particularly the coefficents of each sum is nonzero(because the sets of w_js and v_js are already linear independent), i.e $\exists$ some $c_j \neq 0$ and some $l_k \neq 0$.Then we have $\sum c_j w_j = -\sum l_k v_k \neq 0$,

and the left hand side is in E_i. Therefore consider

$$A\sum c_j w_j \;=\; \lambda_i \sum c_j w_j \;=\; -\lambda_i \sum l_k v_k.$$

$$-A\sum l_k v_k \;=\; -\sum l_k A v_k \;=\; -\sum l_k \lambda_k v_k, \quad \lambda_k \neq \lambda_i.$$

Subtract them,

$$0 \;=\; \sum l_k(\lambda_k - \lambda_i) v_k \;\neq\; 0.$$

However, the right hand side is nonzero, thus it is a contradiction which implies the set $\{w_1, \cdots, w_{r+d}, v_1, \cdots, v_{n-r}\}$ is linear independent. Notice for $\mathbb{R}^n$ space at most it has n linear independent vectors, but we find $n+d$ many linear independent vecotors which is impossible, hence the original assumption is false, that will be $\dim(E_i) \leq r$.
From (i) and (ii), we have $\dim(E_i) = r$, i.e the geometric multiplicity equals to algebraic multiplicity.

□

Remark: If a and b are linear independent, b and c are linear independent, and a and c are linear independent, this does not imply $\{a, b, c\}$ is a linear independent set. Therefore what I presented in my section to show the set is linear independent is incorrect, one should look at the set as a whole.

5.4.46

Given $A = U\Sigma V^T$, we have the spectral decomposition

$$A^T A \;=\; V(\Sigma^T\Sigma)V^T \;=\; V(\Sigma^T\Sigma)V^T.$$

$$AA^T \;=\; U(\Sigma\Sigma^T)U^T \;=\; U(\Sigma\Sigma^T)U^T.$$

5.4.56

Here I present two ways to show this problem.

Proof.

Method 1
According to Schur's decomposition theorem 5.4.11, any matrix $A = UTU^\star$, where U is unitary and T is upper triangular. It is not hard to see Schur's decomposition preserves eigenvalues, i.e matrix T has the same eigenvalues of A. For upper triangular matrix, its eigenvalues are on diag and its determinant equals

to the product of diag elements. Thus

$$\det(A) \;=\; \det(UTU^{\star}) \;=\; \det(U)\det(T)\det(U^{\star}) \;=\; \det(UU^{\star})\det(T) \;=\; \det(I)\det(T) \;=\; \Pi(\lambda_i).$$

Method 2

Consider the characteristic polynomial $\det(\lambda I - A) = P(\lambda) = 0$. By Fundamental theorem of algebra, we can write $P(\lambda) = \Pi(\lambda - \lambda_i)^{r_i}$,where λ_i are the roots of $P(\lambda)$ and r_i are the geometric multiplicity w.r.s.p to each root λ_i. Set $\lambda = 0$, we have

$$(-1)^n \det(A) \;=\; \det(-A) \;=\; P(0) \;=\; \Pi(-\lambda_i)^{r_i} \;=\; (-1)^n\Pi(\lambda_i).$$

Hence, $\det(A) = \Pi\lambda_i$.

□