5.3.6

(i) Find eigenvalues.
Consider
\[ \det(A - \lambda I) = \lambda^2 - 11\lambda + 17 = 0 \]
Apply quadratic formula one can easily find two roots \( \lambda_1 = 9.1401 \) and \( \lambda_2 = 1.8599 \).

(ii) Find dominant eigenvector.
Dominant eigenvector is the eigenvector w.r.s.p to \( \lambda_1 \). Look at \((A - \lambda_1)v = 0\), one has \( v = c(1 \ 0.1401) \), WLOG pick \( c = 1 \) which gives us the same result of example 5.3.5.

(iii) Calculate convergent ratio \( \|q_{j+1} - v\|/\|q_j - v\| \).
Skipped.

(iv) Why \( \|q_{j+1} - v\|/\|q_j - v\| = |\lambda_2/\lambda_1| \).
Since for two by two matrix, it has only two eigenvalues. Refer to the prove of the error convergent ratio estimator on the bottom of page 314, it is exactly \( |\lambda_2/\lambda_1| \).

(v) Why \( \|q_{j+1} - v\|/\|q_j - v\| \neq |\lambda_2/\lambda_1| \) if matrix is larger.
The reason is that refer to the same prove on page 314, for larger matrix, the ratio \( |\lambda_2/\lambda_1| \) becomes the upper bound of exact convergent ratio.

5.3.17

(a)
The process of Shift-and-Invert Strategy for power method, i.e compute \( q_{j+1} \) from \( q_j \), one need do
(i): $\hat{q}_{j+1} = (A - \rho I)^{-1}q_j$.

(ii): Pick $s_{j+1} = \|\hat{q}_{j+1}\|_\infty$, i.e the maximum abs element of $\hat{q}_{j+1}$.

(iii): $q_{j+1} = \hat{q}_{j+1}/s_{j+1}$.

Proceed this process from $q_0$ for a few steps.

(b) 

To calculate the eigenvalues and dominant eigenvector of $A$, do the exactly same calculation as problem 5.3.6, and compare the convergence ratio $\|q_{j+1} - v\|/\|q_j - v\|$ with theoretical convergence ratio $|\lambda_1 - 8|/(\lambda_2 - 8)$.

5.4.5

Let $D = \text{diag}(d_1, d_2, \ldots, d_n)$ and $v_i = (0, 0, \ldots, 1, 0 \ldots)^T$ where the $i$th element is one. Since

$$Dv_i = d_iv_i, \quad \forall i = 1, 2, \ldots, n.$$ 

It implies each diag element $d_i$ is an eigenvalue with the eigenvector $v_i$. Clearly, those $v_i$s are linear independent for $i = 1$ to $n$, thus $n$ linear independent eigenvalues are found.

5.4.25

Proof.

Since similarity transformation preserves the eigenvalues, which means for $B = S^{-1}AS$, if $\lambda_i$ is an eigenvalue of $A$, then it is one of $B$ as well. Thus, this problem is just to show the Dim of eigenspace w.r.s.p to $\lambda_i$ of $A$(Denote this eigenspace as $E_i(A)$) equals to the Dim of eigenspace w.r.s.p to $\lambda_i$ of $B$(Denote this eigenspace as $E_i(B)$). Assume $\dim(E_{i(A)}) = d$, i.e any eigenvector w.r.s.p to $\lambda_i$ of $A$, say $v$ has $v \in \text{span}\{w_1, \ldots, w_d\}$, where $\{w_1, \ldots, w_d\}$ is the basis of $E_{i(A)}$, and can be written in linear combination form $v = \sum_{j=1}^d c_jw_j$.

Now show the set $\{S^{-1}w_j\}_{j=1, \ldots, d}$ is a linear independent set.

Consider

$$\sum c_jS^{-1}w_j = S^{-1}\sum c_jw_j = 0,$$

since $S^{-1}$ is nonsingular, thus the above equation equals to zero is equivalent to $\sum c_jw_j = 0$, by the linear independency of $w_j$s, one has the coefficients $c_j$s are 0. Hence set $\{S^{-1}w_j\}_{j=1, \ldots, d}$ is a linear independent set.

Claim $E_{iB} = \text{span}\{S^{-1}w_j\}_{j=1, \ldots, d}$.

(i) Show $E_{iB} \subset \text{span}\{S^{-1}w_j\}_{j=1, \ldots, d}$.

Pick any element $v^* \in E_{iB}$, one has $Bv^* = \lambda_i v^*$. Since $SB = AS$, then $SBv^* = \lambda_i(Sv^*) = A(Sv^*)$, i.e
$S v^* \in E_{iA}$, thus it can be written as $S v^* = \sum c_j w_j$, and obviously $v^* = S^{-1} \sum c_j w_j = \sum c_j (S^{-1} w_j) \in \text{span}\{w_j\}_{j=1,\ldots,d}$.

(ii) Show $E_{iB} \supset \text{span}\{S^{-1} w_j\}_{j=1,\ldots,d}$.

For any $v^* = \sum c_j (S^{-1} w_j)$, we have $B v^* = BS^{-1} \sum c_j w_j = S^{-1} A \sum c_j w_j = \lambda_i v^*$, thus it is in $E_{iB}$.

All for these two points, we show our claim, and it implies $\dim(E_{iB}) = d$.

\[ \square \]

5.4.26

If $A$ is semisimple matrix, then its eigenvalue has geometric multiplicity equals to its algebraic multiplicity.

Proof.

Assume the algebraic multiplicity of $\lambda_i$ is $r$, i.e there are $r$ eigenvalues equals to $\lambda_i$. We denote the eigenspace w.r.s.p to $\lambda_i$ as $E_i$ that the geometric multiplicity of $\lambda_i = \dim(E_i)$

(i) Show $\dim(E_i) \geq r$.

By theorem (5.4.6), one can decomposite matrix as

$$A = V^{-1} \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_i & \\ & & & \lambda_{i+1} \end{bmatrix} [v_1 \cdots v_n],$$

where the multiple eigenvalue $\lambda_i$ appears $r$ times on the diag of $D$ and all the column of $V$ from $v_i$ to $v_{i+r-1}$ are linear independent eigenvectors w.r.s.p to $\lambda_i$, thus there are at least $r$ linear independent vectors in $E_i$, i.e $\dim(E_i) \geq r$.

(ii) Show $\dim(E_i) \leq r$.

Assume $\dim(E_i) = r + d > r$, and the basis of $E_i$ is $\{w_j\}_{j=1,\ldots,r+d}$. Recall that those columns of $V$ except from $v_i$ to $v_{i+r-1}$ are eigenvectors corresponding to other eigenvalues $\lambda_j \neq \lambda_i$, for simplicity, reindex them as $\{v_1, \ldots, v_{n-r}\}$. Now show the set $\{w_1, \ldots, w_{r+d}, v_1, \ldots, v_{n-r}\}$ is a linear independent set. Consider $\sum c_j w_j + \sum l_k v_k = 0$, use contradiction and let this set be linear dependent, then one has a nonzero solution for those coefficients and particularly the coefficients of each sum is nonzero (because the sets of $w_j$s and $v_j$s are already linear independent), i.e $\exists$ some $c_j \neq 0$ and some $l_k \neq 0$. Then we have $\sum c_j w_j = - \sum l_k v_k \neq 0$, 

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and the left hand side is in $E_i$. Therefore consider
\[ A \sum c_j w_j = \lambda_i \sum c_j w_j = -\lambda_i \sum l_k v_k. \]
\[ -A \sum l_k v_k = -\sum l_k A v_k = -\sum l_k \lambda_k v_k, \quad \lambda_k \neq \lambda_i. \]
Subtract them,
\[ 0 = \sum l_k (\lambda_k - \lambda_i) v_k \neq 0. \]

However, the right hand side is nonzero, thus it is a contradiction which implies the set \{w_1, \cdots, w_{r+d}, v_1, \cdots, v_{n-r}\} is linear independent. Notice for $\mathbb{R}^n$ space at most it has $n$ linear independent vectors, but we find $n + d$ many linear independent vectors which is impossible, hence the original assumption is false, that will be $\dim(E_i) \leq r$.

From (i) and (ii), we have $\dim(E_i) = r$, i.e the geometric multiplicity equals to algebraic multiplicity.

Remark: If $a$ and $b$ are linear independent, $b$ and $c$ are linear independent, and $a$ and $c$ are linear independent, this does not imply $\{a, b, c\}$ is a linear independent set. Therefore what I presented in my section to show the set is linear independent is incorrect, one should look at the set as a whole.

5.4.46

Given $A = U \Sigma V^T$, we have the spectral decomposition
\[ A^T A = V (\Sigma^T \Sigma) V^T = V (\Sigma^T \Sigma) V^T. \]
\[ A A^T = U (\Sigma \Sigma^T) U^T = U (\Sigma \Sigma^T) U^T. \]

5.4.56

Here I present two ways to show this problem.

Proof.

Method 1

According to Schur’s decomposition theorem 5.4.11, any matrix $A = U T U^*$, where $U$ is unitary and $T$ is upper triangular. It is not hard to see Schur’s decomposition preserves eigenvalues, i.e matrix $T$ has the same eigenvalues of $A$. For upper triangular matrix, its eigenvalues are on diag and its determinant equals
to the product of diag elements. Thus

\[
\det(A) = \det(UTU^*) = \det(U) \det(T) \det(U^*) = \det(UU^*) \det(T) = \det(I) \det(T) = \Pi(\lambda_i).
\]

Method 2
Consider the characteristic polynomial \(\det(\lambda I - A) = P(\lambda) = 0\). By Fundamental theorem of algebra, we can write \(P(\lambda) = \Pi(\lambda - \lambda_i)^{r_i}\), where \(\lambda_i\) are the roots of \(P(\lambda)\) and \(r_i\) are the geometric multiplicity w.r.s.p to each root \(\lambda_i\). Set \(\lambda = 0\), we have

\[
(-1)^n \det(A) = \det(-A) = P(0) = \Pi(-\lambda_i)^{r_i} = (-1)^n \Pi(\lambda_i).
\]

Hence, \(\det(A) = \Pi\lambda_i\).