3.12b does not have identity:

Suppose for contradiction that $e \in G$ is the identity. Choose $a \in G$ such
(this is possible since $G$ has no minimal element).
Then $\max \{ a, e \} = e \neq a$ so $e$ is not the identity, a contradiction.
Since there is no identity, elements also cannot have inverses.
(This is associative and closed.)

2d This does not have an identity:

Suppose for contradiction that $e \in G$ is the identity. Consider
$-2 \times e = 1 - 2e \in 1$ by def of $\times$
$= -2$ by def of identity.
This is impossible since $-2 \in G$ and the absolute value of an integer
cannot be negative.
Since there is no identity, elements cannot have an inverse.
(This is associative and closed.)

10. Closure: let $f, g, m_1 x + b_1, \ g, m_2 x + b_2$ be arbitrary elements in $G$ (so $m_1$ and $m_2 \neq 0$).

$f \circ g = f(m_2 x + b_2) + m_1 (m_2 x + b_2) + b_1 = m_1 (m_2 x + b_2) + b_1$.
Since $m_1, m_2, b_1, b_2 \in \mathbb{R}$,
$m_1 m_2 \in \mathbb{R}$ (and $m_1 m_2 \neq 0$ since $m_1 \neq 0$ and $m_2 \neq 0$) and $m_1 b_2 + b_1 \in \mathbb{R}$ so $f \circ g \in G$.
Associativity: This follows from associativity of function composition and won't
always need to be shown. For this problem I will show it.
Let $f$ and $g$ be as above and $h = m_3 x + b_3$ be an arbitrary element of $G$.

$(f \circ g) h = (m_1 (m_2 x + b_2) + b_1) h = m_1 (m_2 (m_3 x + b_3) + b_2) + b_1$.

$f(g h) = f(m_2 (m_3 x + b_3) + b_2) = m_1 (m_2 (m_3 x + b_3) + b_2) + b_1$.
Since these are equal, $\circ$ is associative.

Identity: I claim $e = x$ is the identity function: If $f \in G$ is arbitrary,

$f \circ e = f(x) = m_1 x + b_1 = f$ and
$e \circ f = e(m_1 x + b_1) = m_1 x + b_1 = f$ so $x$ is the identity.

Inverse: Let $f \in G$ be arbitrary and choose $g = \frac{1}{m_1} x - \frac{b_1}{m_1} \in G$ since $m_1 \neq 0$ and

$\frac{1}{m_1} \cdot -b_1 \in \mathbb{R}$.

$f \circ g = f \left( \frac{1}{m_1} x - \frac{b_1}{m_1} \right) = m_1 \left( \frac{1}{m_1} x - \frac{b_1}{m_1} \right) + b_1 = x - b_1 + b_1 = x = e$ and

g \circ f = \frac{1}{m_1} (m_1 x + b_2) - \frac{b_1}{m_1} = x$ so $g = f^{-1}$. Since $f$ was arbitrary, all elements in $G$ have
inverses and since all 4 conditions for a group are met, $G$ is a group.
Closure: Let \( \begin{bmatrix} m_1 & b_1 \\ 0 & 1 \end{bmatrix} \) and \( \begin{bmatrix} m_2 & b_2 \\ 0 & 1 \end{bmatrix} \) be arbitrary elements in \( G \) (so \( m_1, m_2 \neq 0 \)).

\[
\begin{bmatrix} m_1 & b_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 & b_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_1 m_2 & m_1 b_2 + b_1 \\ 0 & 1 \end{bmatrix}
\]
and \( m_1, m_2 \neq 0 \) so this is in \( G \).

Associativity of matrix multiplication was proven in class.

Identity: Let \( e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) and note that \( e \in G \). For an arbitrary element in \( G \),

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 & b_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_1 & b_1 \\ 0 & 1 \end{bmatrix}
\]
and \( \begin{bmatrix} m_1 & b_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_1 & b_1 \\ 0 & 1 \end{bmatrix} \) so \( e \) is the identity.

Inverse: Let \( \begin{bmatrix} m_1 & b_1 \\ 0 & 1 \end{bmatrix} \in G \) be arbitrary and consider \( \begin{bmatrix} m_1^{-1} & -b_1 m_1^{-1} \\ 0 & 1 \end{bmatrix} \) which is also in \( G \).

\[
\begin{bmatrix} m_1 & b_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1^{-1} & -b_1 m_1^{-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -b_1 b_1^{-1} \\ 0 & 1 \end{bmatrix} = e \text{ and}
\]

\[
\begin{bmatrix} m_1^{-1} & -b_1 m_1^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 & b_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \] so elements have inverses and \( G \) is a group.

Closure: Sum and products of real numbers are real and it remains to show \( ab \neq -1 \).

Suppose for contradiction that \( ab = -1 \). Then \( ab + b + 1 = 0 \) and

\[
a(1+b)+b(1) = 0 \quad \text{and} \quad (a+1)(b+1) = 0
\]
which implies \( a = -1 \) or \( b = -1 \), a contradiction.

Associativity: \( (ab)c = (ab)bc = abc + ab + ac + bc \)

\[
a(bc) = (a+b+c) = abc + ab + ac + bc + abc.
\]

Since these are equal and above \( m = 3 \) were arbitrary, associativity holds.

Identity: Let \( e = 0 \), \( a = m = 3 \) arbitrary.

\[
0 + a = a + 0 = a \quad \text{and} \quad a + 0 = a + 0 = a \quad \text{so} \quad 0 \quad \text{is the identity}.
\]

Inverse: (As most naturally calculated by students).

Let \( a \in m = 3 \) arbitrary.

\[
G = a + b = ab + bc
\]

\[
\Rightarrow -a = b(1 + a)
\]

\[
\Rightarrow b = \frac{-a}{1 + a}.
\]

Note that \( b \in G \) (since \( a + 1 \neq 0 \) when \( a \neq -1 \)) and \( b \neq -1 \) since \( -1 = \frac{-a}{1 + a}, \Rightarrow -a - 1 = a \Leftrightarrow 0 = -2 \).

Thus, \( b \in m = 3 \) and we have shown it is a right inverse. We must show it is a left inverse.

\[
b \cdot a = \frac{-a}{1 + a} + a = \frac{-a + a^2}{a + 1} + \frac{-a^2}{a + 1} = \frac{-a + a^2}{a + 1} - \frac{a^2}{a + 1} = 0 \cdot e.
\]

Thus, \( b \) is a two-sided inverse in \( m = 3 \) and so this is a group.
3.15 \[ g^2 = g \]
\[ e = g^{-1}g \]
\[ e = e \]
\[ g = e. \]

22 \( \implies \) Suppose \( G \) is abelian and let \( a \) be \( G \) arbitrary.
\[(ab)^{-1} = b^{-1}a^{-1} \quad (\text{as shown in class})
\]
\[= a^{-1}b^{-1} \quad (\text{since } a^{-1}b^{-1} \in G \text{ and } G \text{ abelian } \implies a^{-1}b^{-1} = b^{-1}a^{-1}).
\]

\( \implies \) Suppose \((ab)^{-1} = a^{-1}b^{-1} \) where \( a, b \in G \) are arbitrary.
\[ab = ((ab)^{-1})^{-1} \quad (\text{by properties of inverse})
\]
\[= (a^{-1}b^{-1})^{-1} \quad (\text{since } (ab)^{-1} = a^{-1}b^{-1} \text{ by assumption})
\]
\[= (b^{-1})^{-1}(a^{-1})^{-1} \quad (\text{as shown in class})
\]
\[= ba.
\]
So \( G \) is abelian.

Extra 1: Order: I claim that \( G \) as a set equals \( \{1, e^{2\pi i k/n}, e^{4\pi i k/n}, \ldots, e^{2(n-1)\pi i k/n} \} \).

To show this, let \( g \in G \), \( k \in \mathbb{Z} \) be an arbitrary element of \( G \).

By the division algorithm, \( k = bn+r \), \( b, r \in \mathbb{Z}, 0 \leq r < n-1 \).

Then \( g = e^{2\pi i (bn+r)/n} = e^{2\pi i kn/n + 2\pi i r/n} = e^{2\pi i r/n} \) since \( e^{2\pi i k/n} \) is an element in the set listed above and since \( g \) was arbitrary, any element of \( G \) is equal to \( 1 \) of the elements listed above.

Conversely, note that every element listed in the set above is actually in \( G \).

It remains to show that the \( n \) elements in the above set are distinct.

Suppose we have 2 arbitrary elements from the above set: \( e^{2\pi i k/n}, e^{2\pi i j/n} \), \( 0 \leq k, j \leq n-1 \) and suppose \( e^{2\pi i j/n} = e^{2\pi i k/n} \). Then \( e^{2\pi i j/n - 2\pi i k/n} = 1 \) which implies \( n \mid j - k \).

So \( j = k \) for some \( k \in \mathbb{Z} \). \( 0 \leq k, j \leq n-1 \) so \(- (n-1) \leq k-j \leq n-1 \) and this implies that \( a = 0 \) and \( k = j \). Thus, the only way two elements in the set above are equal is if the exponents are equal and so there are \( n \) distinct elements: \( \{e^{2\pi i k/n} \mid 0 \leq k \leq n-1 \} \).
(Extra 1) Group Closure: Let \( e^{2\pi i n}, e^{2\pi i k} \) be arbitrary elements in \( G \).

\[
e^{2\pi i n} \cdot e^{2\pi i k} = e^{2\pi i (n+k)l} = e^{2\pi i} = e
\]

and this is in \( G \) since \( j+k \in \mathbb{Z} \).

Associativity follows from associativity of complex numbers but can also be shown:

\[
(e^{2\pi i n}, e^{2\pi i k}) \cdot e^{2\pi i l} = e^{2\pi i (n+l)} = e^{2\pi i (j+k+l)} = e^{2\pi i (j+k+k')} = e^{2\pi i (j+k+l)}
\]

Identity: Let \( \text{id} = e^{2\pi i (0)} = e^{0} = 1 \). Note that \( 1 \in G \) (taking \( k=0 \) in \( e^{2\pi i k} \)).

\[
e^{2\pi i n} \cdot 1 = e^{2\pi i n} \quad \text{and} \quad 1 \cdot e^{2\pi i n} = e^{2\pi i n} \quad \text{so} \quad 1 \text{ is the identity.}
\]

Inverse: Let \( e^{2\pi i k/n} \) be arbitrary and consider \( e^{2\pi i (-k)/n} \) which is also in \( G \).

\[
e^{2\pi i k/n} \cdot e^{2\pi i (-k)/n} = e^{2\pi i (k-k)/n} = e^{0} = 1 = \text{id}
\]

so elements have inverses and this is a group.

Extra 2 (Given associativity, \( e \cdot e = e \cdot e = e \), \( \forall a \in G \) so \( e \) is the \( (2\text{-sided}) \) identity.

Let \( a \in G \) arbitrary and \( b \) be such that \( ba \cdot e = e \).

Then \( bae = ba(e) = ba(a) = eba \) (since \( ba \cdot e \) is given)

= \( eba \) (since \( ba \cdot e \) is given)

= \( ba \) (since \( e \cdot b \cdot a \) is given).

So \( ba = e \) and (by applying the left inverse of \( e \), which exists, to both sides) \( c = a \).

Since \( c \cdot a = e \cdot e = a \) \( \forall a \in G \) so \( e \) is the \( (2\text{-sided}) \) identity.

Let \( a \in G \) arbitrary and \( e \) be such that \( ba = e \).

Then \( ab = eab \)

\[
= a(ba)b = (ab)ab.
\]

Since \( ab \in G \), it has a left inverse. Multiplying both sides by this gives \( e = ab \).

Thus, \( b \) is the right-inverse of \( a \), inverses exist, and so this is a group.