Research Statement

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I study noncommutative algebra, specifically Ore extensions and Artin-Schelter regular algebras with connections to noncommuatative algebraic geometry.

AS-regular algebras are noncommutative polynomial rings that in some sense generalize commutative polynomial rings while maintaining some very "nice" properties, such as having finite dimension. In particular, these properties allow AS-regular algebras to be used to construct a noncommutative equivalent of projective schemes. Research in noncommutative algebraic geometry and its applications to fields such as mathematical physics relies heavily on analyzing specific examples of quantum $\mathbb{P}^n$'s, which can be constructed algebraically by forming the noncommutative projective scheme $\mathrm{Proj}(A)$ where A is a noetherian AS-regular algebra of global dimension $n+1$. Thus, the classification of AS-regular algebras, and less generally the explicit construction of examples of such algebras, is an extremely active area of current research in the field.

Under mild assumptions, iterated Ore extensions are AS-regular algebras with additional "nice" properties. For example, they have the same K-vector space basis as commutative polynomial rings, $\{x_1^{i_1} \cdots x_n^{i_n}\}$, and multiplication in these algebras is somewhat well understood. As a result, their classification provides a natural starting place for the classification of AS-regular algebras in higher dimensions, which remain very poorly understood in general.

1. Background

Definition 1.1. Let R be a ring. An *Ore extension* $R[x, \sigma, \delta]$ is a ring with elements of the form $f(x) = \sum_{i=0}^{n} a_i x^i, a_i \in R$ and multiplication satisfying $xr = \sigma(r)x + \delta(r)$ for all $r \in R$ where σ is an endomorphism of R and δ is a σ-derivation of R, i.e. $\delta(r_1 r_2) = \sigma(r_1)\delta(r_2) + \delta(r_1)r_2$ for all $r_1, r_2 \in R$.

An *iterated Ore extension* $R[x_1, \sigma_1, \delta_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$ is an Ore extension where for all $j > 1$, σ_j and δ_j are a ring endomorphism and a σ_j-derivation of $R[x_1, \sigma_1, \delta_1][x_2, \sigma_2, \delta_2] \cdots [x_{j-1}, \sigma_{j-1}, \delta_{j-1}]$, respectively. Elements in this extension have the form $\sum a_i x_1^{i_1} \cdots x_n^{i_n},\ a_i \in R$.

Let K be an algebraically closed, characteristic 0 field and A an $\mathbb{N}$-*graded* K-algebra, $A = \bigoplus_{i=0}^{\infty} A_i$ with $A_i A_j \subseteq A_{ij}$ for all i and j. An iterated Ore extension $K[x_1][x_2, \sigma_2, \delta_2] \cdots [x_n, \sigma_n, \delta_n]$ which is also graded and has every σ_i injective is also AS-regular as defined below [AST, Proposition 2].

While this definition provides a convenient way to present an Ore extension in terms of the multiplication, it is often also useful to present the algebra as the quotient of a free algebra by a finitely generated homogeneous ideal.

Example 1.2. Consider the graded iterated Ore extension A with variables (x_3, x_2, x_1) of respective degrees (1,1,2) defined by the relations:

$$\begin{aligned} r_{32} &: x_3 x_2 = x_1 + x_2 x_3 \\ r_{31} &: x_3 x_1 = x_1 x_3 \\ r_{21} &: x_2 x_1 = x_1 x_2. \end{aligned}$$

In order for the multiplication to be a well-defined associative operation, any choice of multiplication should lead to the same result. In particular, it is necessary to compute the overlap:

$$\begin{aligned}(x_3x_2)x_1 &= (x_1 + x_2x_3)x_1 \\ &= x_1x_1 + x_2(x_1x_3) \\ &= x_1x_1 + x_1x_2x_3, \text{ while} \\ x_3(x_2x_1) &= x_3(x_1x_2) \\ &= (x_1x_3)x_2 \\ &= x_1(x_1 + x_2x_3) \\ &= x_1x_1 + x_1x_2x_3.\end{aligned}$$

Since these are equal, this is actually an Ore extension and every element can be written in the form $\sum a_i x_1^{i_1} x_2^{i_2} x_3^{i_3}$.

Alternatively, r_{32} can be solved for x_1 and this algebra can be viewed as something generated by the degree 1 variables x_3 and x_2:

$$\begin{aligned}[r_{32} :&x_1 = x_3x_2 - x_2x_3] \\ r_{31} :&x_3(x_3x_2 - x_2x_3) = (x_3x_2 - x_2x_3)x_3 \\ &x_3x_3x_2 = 2x_3x_2x_3 - x_2x_3x_3 \\ r_{21} :&x_2(x_3x_2 = x_2x_3) = (x_3x_2 - x_2x_3)x_2 \\ &x_3x_2x_2 = 2x_2x_3x_2 - x_2x_2x_3.\end{aligned}$$

Then $A \cong \dfrac{K\langle x_2, x_3\rangle}{\langle x_3x_3x_2 = 2x_3x_2x_3 - x_2x_3x_3,\ x_3x_2x_2 = 2x_2x_3x_2 - x_2x_2x_3\rangle}$.

A is *generated in degree 1* since it can be expressed as the quotient of a free algebra with degree 1 variables and it has *relation type* (3,3) since the ideal of the quotient is minimally generated by 2 relations, each of degree 3.

Although not necessary to understand my results, it is also worth briefly presenting the background and history of AS-regular algebras.

Let K be an algebraically closed, characteristic 0 field and $A = \bigoplus_{i=0}^{\infty} A_i$. A is called *connected* if $A_0 = K$ and has *Hilbert series* $h_A(t) = \sum_{i=0}^{\infty} dim_k(A_i)t^i$.

Definition 1.3. A connected graded K-algebra A is *Artin-Schelter (AS) regular* of dimension d if

(1) A has finite global dimension d,
(2) A has finite GK dimension, and
(3) $Ext^i(k_A, A) \cong \begin{cases} 0 & i \neq d \\ K(l) & i = d. \end{cases}$

An AS-regular algebra of dimension 2 which is generated in degree 1 is isomorphic to either the Jordan plane $J = \dfrac{K\langle x_1, x_2\rangle}{\langle x_2x_1 - x_1x_2 - x_1^2\rangle}$, or a quantum plane $\mathcal{O}_q = \dfrac{K\langle x_1, x_2\rangle}{\langle x_2x_1 - qx_1x_2\rangle}$. The possible families of relations of AS-regular algebras of dimension 3 which are generated in degree 1 were completely classified by Artin, Tate, and Ven den Bergh [AS],[ATVdB].

The classification of AS-regular algebras of dimension 4 remains an active area of research. Restricting to AS-regular algebras which are domains and generated in degree 1, the possible *relation types* i.e. the number and degrees of the minimal set of relations generating the ideal are known. If the algebra is assumed to be $\mathbb{Z}^2$*-graded*, i.e. each generator has degree $(1,0)$ or $(0,1)$ and each relation is $\mathbb{Z}\times\mathbb{Z}$-homogeneous, then the possible families of relations are known in most cases [LPWZ], [RZ], [ZZ2].

A number of interesting patterns have arisen for AS-regular algebras. For any possible relation type of an algebra of dimension 4 or less, the Hilbert series of the algebra is unique, there is an *enveloping algebra* of a graded Lie algebra with the given relation type, and there is a $\mathbb{Z}^2$-graded algebra with the given relation type.

Although the classification of AS-regular algebras of dimension 5 is also an active area of research, progress in the area has been slow. In 2011, Floystad and Vatne listed the possible relation types of an AS-regular algebra of dimension 5 with 2 degree 1 generators under mild assumptions and provided an example of an AS-regular algebra with a relation type that could not possibly be realized by an enveloping algebra [FV]. Building on their work, Wang and Wu used A_∞ techniques to find many families of algebras of dimension 5 with two generators, including an Ore extension with 3 degree 4 relations and 2 degree 5 relations, i.e. relation type (4,4,4,5,5) [WW]. This relation type provides another example of something that cannot be realized by an enveloping algebra and is the first example where algebras with the same Hilbert series can have different resolution types. There has not yet been a careful treatment of the classification of dimension 5 algebras with 3 or 4 generators.

2. My Research

My dissertation research has focused on the classification of possible relation types of dimension 5 AS-regular algebras which are also graded iterated Ore extensions generated in degree 1 and which have all σ_i automorphisms. For brevity, I refer to such algebras as *AS-Ore* extensions. I began by writing a presentation for graded iterated Ore extensions and then rewrote the possible algebras in terms of their degree 1 generators in order to study their possible relation types.

Theorem 2.1. *If K is a field and $P = K[x_1][x_2,\sigma_2,\delta_2]\cdots[x_n,\sigma_n,\delta_n]$ is a graded iterated Ore extension, then P has presentation*

$$P \cong \frac{K\langle x_1\cdots x_n\rangle}{\langle\{r_{ji}\}\rangle}$$

where for each $j > i$, there is a unique homogeneous relation r_{ji}, given by

$$x_j x_i = \sigma_j(x_i)x_j + \delta_j(x_i),\ \sigma_j(x_i) \text{ and } \delta_j(x_i) \in K[x_1]\cdots[x_{j-1},\sigma_{j-1},\delta_{j-1}],$$

and these relations satisfy the diamond condition.

Here,"satisfying the diamond condition" is equivalent to checking that the multiplication is well defined, i.e. that the value of overlaps $x_k x_j x_i$, $x_k > x_j > x_i$ is independent of which relation is used to simplify the expression.

For an Ore extension with variables of degrees 1, 1, 2, 3, and 5, I used this presentation to write fully general relations that meet the requirements on the σ_i

and δ_i and then used computer software to compute the overlaps $(x_kx_j)x_i - x_k(x_jx_i)$ for all $x_k > x_j > x_i$. Setting each coefficient in these overlaps equal to zero provided me with a large system of equations. Although this system of equations was too complex for the computer to solve, I was able to simplify it substantially by focusing on the most informative equations and later by setting some coefficients equal to zero. Using this method I was able to prove:

Theorem 2.2. *The following relations define an AS-Ore extension which has* $h(t) = \dfrac{1}{(1-t)^2(1-t^2)(1-t^3)(1-t^5)}$, *relation type (3,4,7), and variables of degrees 1, 1, 2, 3, and 5:*

$$\begin{aligned}
r_{21} &: x_2x_1 = -x_1x_2 \\
r_{32} &: x_3x_2 = x_1 + bx_2x_3 \\
r_{31} &: x_3x_1 = -x_1x_3 \\
r_{43} &: x_4x_3 = x_2 + bx_3x_4 \\
r_{42} &: x_4x_2 = b^2x_2x_4 \\
r_{41} &: x_4x_1 = x_1x_4 \\
r_{54} &: x_5x_4 = x_3 + x_4x_5 \\
r_{53} &: x_5x_3 = -x_3x_5 \\
r_{52} &: x_5x_2 = -x_2x_5 - b^2x_3x_3 \\
r_{51} &: x_5x_1 = x_1x_5 + cx_3x_3x_3
\end{aligned}$$

where $b = e^{\frac{4\pi i}{3}}$ *and* $c = \frac{2b^2}{1-b+b^2}$.

This theorem is of interest because this is the Hilbert series of the algebra found by Floystad and Vatne for which there is no enveloping algebra with the same series. Together with other results in the field, this means that every known relation type of AS-regular algebra can be realized by an AS-Ore extension. This lends some support to the hypothesis that every relation type of AS-regular algebra (or at least of every AS-regular algebra of dimension at most 5) can be realized by an AS-Ore extension, although more examples will be required before there is reason to suspect that such a general statement holds.

I also investigated the classification of relation types of AS-Ore extensions of dimension 5 with 3 and 4 generators. The free resolution of the trivial module K for a dimension 5 AS-regular algebra generated in degree 1 is well known and can be used to find the Hilbert series of the algebra. On the other hand, the Hilbert series of an iterated Ore extension is the same as that of a weighted commutative polynomial ring with variables of the same degrees and so is equal to $\dfrac{1}{\prod(1-t^{deg(x_i)})}$. By comparing these series, I was able to restrict the possible relation types of AS-Ore extensions. I was able to further restrict the relation types by seeing what happened to the relations given by Theorem 2.1 after writing everything in terms of the degree 1 generators. These methods allowed me to completely classify the relation types of AS-Ore extensions of dimension 5 with 3 and 4 generators. The results can be summarized as:

Theorem 2.3. *An AS-Ore extension of dimension 5 with 3 generators has relation type (2,2,3), (2,2,3,4), or (2,3,3,3,3,3).*

Theorem 2.4. *An AS-Ore extension of dimension 5 with 4 generators has relation type (2,2,2,2,2), (2,2,2,2,2,3), or (2,2,2,2,2,3,3).*

Motivated by previous results in the field and the example presented by Floystad and Vatne, I also investigated the possible relation types of enveloping algebras of graded Lie algebras, a subset of AS-Ore extensions. For each relation type listed above, I was either able to construct explicit examples of enveloping algebras or use a general presentation of enveloping algebras to conclusively show that no such enveloping algebra could exist. I summarize the results as:

Theorem 2.5. *An AS-Ore extension of dimension 5 which has 3 or 4 generators and which is also the enveloping algebra of a graded Lie algebra has relation type (2,2,2,2,2), (2,2,2,2,2,3,3), (2,3,3,3,3,3), or (2,2,3,4).*

I have also explicitly constructed an example of an AS-Ore algebra for each of the relation types listed in Theorem 2.3 and Theorem 2.4.

3. Remaining questions and future directions

There are several loose ends from my dissertation research that would be interesting to revisit at a later date. From the work of authors that have looked at AS-regular algebras of dimension 5 with 2 generators, together with my own results, we have a complete classification of the possible relation types of AS-Ore extensions of dimension 5, with just one exception:

Question 3.1. *Is there an AS-Ore extension with relation type (4,4,4,5)?*

This relation type has variables of degrees 1, 1, 2, 3, and 3. If the variables $(x_5, x_4, x_3, x_2, x_1)$ are taken to have degrees (1,1,2,3,3) and if I investigate all possible AS-Ore extensions of the form $K[x_1][x_2, \sigma_2, \delta_2][x_3, \sigma_3, \delta_3][x_4, \sigma_4, \delta_4][x_5, \sigma_5, \delta_5]$ then my initial computations suggest that the algebra may have relation type (4,4,4) or (4,4,4,5,5), but not (4,4,4,5). I have also run computations where the variables are adjoined in different orders, but have not yet checked or exhausted these cases. It appears likely that, in all cases, there are relations of degrees 4, 4, 4, 5, and 5 (and that the leading terms of these relations are always the same), but that some of the degree 5 relations may be consequences of overlaps that fail to resolve when the algebra is written in terms of the degree 1 generators. In this case, these relations are not part of the minimal generating set of the ideal. It also appears likely that either none or both of the degree 5 relations are independent of overlaps, so that (4,4,4,5) is not a possible relation type. Finishing these computations would be a highly accessible problem, and much of the work would be appropriate for an undergraduate student interested in the field.

More generally, it is appropriate to ask:

Question 3.2. *Is there an AS-regular algebra generated in degree 1 with relation type (4,4,4,5)?*

Such an algebra would be quite interesting as it would represent the first example of a relation type which cannot be realized by any $\mathbb{Z}^2$-graded AS-regular algebra. (Zhou and Lu found that there is no $\mathbb{Z}^2$-graded algebra with this relation type [ZL].) It would also likely be an example of a relation type which cannot be realized by any

AS-Ore extension. An algebra with this relation type would still have 3 relations of degree 4 in the minimal generating set of the ideal, but there would be much greater flexibility in their leading terms and the number and degrees of additional relations.

A question which has historically been asked about AS-regular algebras of lower dimensions is whether or not they are PI. An algebra A is called *polynomial identity* if there is an N and a nonzero polynomial P in N noncommuting variables with coefficients in $\mathbb{Z}$ such that $P(a_1, \cdots, a_N) = 0$ for any $a_1, \cdots, a_N \in A$. Any commutative algebra is PI since it satisfies $P = X_1X_2 - X_2X_1$. Thus, PI algebras are of interest since they are close generalizations of commutative algebras. In lower dimensions, the majority of AS-regular algebras are not PI, but every relation type can be realized by an algebra that is.

It is reasonable to ask:

Question 3.3. *For which of the relation types found is there an AS-Ore extension which is PI? Are there any for which there is no AS-Ore extension which is not PI?*

Leroy and Matczuk have proven that, if R is PI, σ injective, and the center of $R[x, \sigma, \delta] \cap \langle x \rangle$ is not trivial, then $R[x, \sigma, \delta]$ is also PI [LM, Theorem 2.7]. I believe that this can be used recursively to show that the example of an AS-Ore extension given in Theorem 2.2 is PI. On the other hand, an enveloping algebra which is generated in degree 1 and which is not already commutative is not PI [Pas, Theorem 1.3]. Although I have not carefully examined which of the relation types of AS-Ore extensions with 3 and 4 generators can be realized by algebras which are and are not PI, this would be another interesting and reasonably accessible question since it can largely be reduced to examining central elements of the given algebras. Large portions of this project would be appropriate for a talented undergraduate student, especially one with interest in programming.

Given the direction that other experts have gone in the field, it is also natural to ask about AS-regular Ore extensions that are $\mathbb{Z}^2$-graded. A first question in this area could be:

Question 3.4. *For which of the relation types found is there an AS-Ore extension which is also $\mathbb{Z}^2$-graded?*

The example of an AS-Ore extension with 2 generators in Theorem 2.2 is $\mathbb{Z}^2$-graded, as are many of the enveloping algebras I've found. On the other hand, the example I have found of an Ore extension with relation type (2,2,2,2,2,3) is not. In fact, I noted that I could get the correct relation type by setting a specific coefficient from the general relations equal to 1, and this coefficient makes the algebra fail to be $\mathbb{Z}^2$-graded. While I have not exhausted alternate approaches that may give the correct relation type while maintaining the grading, this first example suggests that the question is worth investigating and may have a very surprising answer.

$\mathbb{Z}^2$-graded algebras are interesting outside of the context of AS-Ore extensions and are an appropriate starting point for generalizing the results of my dissertation research. It is natural to ask:

Question 3.5. *Is it possible to classify all families of $\mathbb{Z}^2$-graded AS-regular algebras which have the same relation types as those that have been found for AS-Ore extensions?*

In the coming months, I hope to quickly address Question 3.4 before switching gears to answer Question 3.5. Previous results in the field will prove useful here. For example, Rogalski and Zhang classified most $\mathbb{Z}^2$-graded AS-regular algebras of dimension 4 with 3 generators. They found that the degree 2 relations must have grading (2,0) and (1,2), a fact which carries over to algebras of dimension 5 with 3 generators if the authors' initial assumptions are maintained. The possible degree 2 relations can be specified even further, and a great deal is also known about the degree 3 relation. In the dimension 5 case however, there are a number of other potential relations about which very little is known.

Widening the scope even more, one could ask:

Question 3.6. *Are there $\mathbb{Z}^2$-graded AS-regular algebras with the same Hilbert series as those found but with different relation types?*

Question 3.7. *Are there $\mathbb{Z}^2$-graded AS-regular algebras of dimension 5 with Hilbert series different from those found?*

Question 3.8. *Are there (not necessarily $\mathbb{Z}^2$-graded) AS-regular algebras of dimension 5 with relation type or Hilbert series different from those found?*

Answering Question 3.8 remains an extremely difficult project beyond the scope of methods that have been developed and used in the field so far, but answering the preceding questions should help to guide our intuition. Additionally, generalizations of techniques such as that used to prove [FV, Theorem 5.6] may allow us to list or restrict the possible relation types of algebras of dimension 5, which would be a very interesting result in its own right and a helpful tool in answering Question 3.8.

There are several other generalizations that might be of interest in broader exploration of AS-regular algebras. A question that would generalize my results on AS-Ore extensions and provide additional examples of dimension 5 AS-regular algebras for us to work with is:

Question 3.9. *What are the possible relation types of AS-regular algebras of dimension 5 that are* double Ore extensions *as defined in* [ZZ1]*?*

This could be a long term project appropriate for a group of undergraduate students. While understanding the construction of a double Ore extension and the theory behind it would require a moderate investment of time initially, it would then be reasonable for a student to find a presentation for double Ore extensions similar to that given for Ore extensions in Theorem 2.1 and to then use similar computational methods to explore possible relations.

Alternatively, there are questions that could be asked about *iterated Hopf Ore extensions*, which are both iterated Ore extensions and Hopf algebras. Any enveloping algebra is automatically an IHOE with $\triangle(x_i) = x_i \bigotimes 1 + 1 \bigotimes x_i$ for every generator. A starting point for research in this area would be the question:

Question 3.10. *Can every relation type found be realized by an IHOE?*

This question is especially interesting since we now have many examples in dimension 5 of relation types that cannot be realized by enveloping algebras. It made sense to generalize enveloping algebras to less restrictive objects, but there was no clear reason to choose Ore extensions over IHOE's, which lie between Ore extensions and enveloping algebras. It would be interesting to know if IHOE's are "as flexible" as Ore extensions in the possible relation types they can satisfy or if, like enveloping algebras, they are in some way overly restrictive.

Finally, there is an extensive pool of questions that have historically been of interest to mathematicians and physicists who study AS-regular algebras and which have motivated their classification and the search for additional examples.

For example, A *point module* of an AS-regular algebra A generalizes the concept of a point in Proj(A) and is an A-module $M = \sum_{i=0}^{\infty} M_i$ with $dim(M_i) = 1$ and which is generated by M_0. In lower dimensions, the point modules of an algebra frequently form a "nice" space which has proven useful to study, as has the ideal which kills all point modules of A. There is some evidence to suggest that point modules are quite rare and difficult to compute in higher dimensions and the additional examples of dimension 5 algebras which I have found would allow us to explore that possibility further.

As another example, another active area of inquiry deals with the generalization of classical invariant theory to the noncommutative setting. If A is AS-regular and G is a finite group which acts on A, let $A^G = \{a \in A \mid g(a) = a \ \forall g \in G\}$. We can ask if A^G is again AS-regular or at least *AS-Gorenstein* (where we drop the condition that global dimension be finite).

Question 3.11. *What is $Aut(A)$ for the examples of AS-Ore extensions I have found?*

More generally, we can ask these same questions about A^H where H is a Hopf algebra which acts on A:

Question 3.12. *Are there any interesting Hopf-actions on A for the examples found?*

This field provides another motivation for the explicit construction of additional examples of AS-regular algebras as they are required to test and further refine existing conjectures about invariant theory.

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