• Determine $7^{2021} \mod 5$.

• Show that $5n + 3$ and $7n + 4$ are relatively prime for all $n$.

• For any positive integer $k$, find $N$ such that the integers $N + 1, \ldots, N + k$ are all composite.

• For any positive integer $N$ denote $a_\ell a_{\ell-1} \ldots a_1 a_0$ as its decimal expansion. For example $N = 341$ implies $\ell = 2$, $a_2 = 3$, $a_1 = 4$ and $a_0 = 1$. Define digit-wise sum on the set of positive integers, denote it by $\oplus$, of two decimal numbers $N = a_\ell \ldots a_1 a_0$ and $M = b_r \ldots b_1 b_0$ given by

$$N \oplus M = c_s \ldots c_1 c_0$$

where $c_i$ is the digit satisfying $c_i \equiv a_i + b_i \mod 10$ (there is no carry!). Here $s = \max(\ell, r)$ and digits are taken to be zero when not defined. For example $341 \oplus 94 = 335$.

– Show that $\oplus$ is an associative operation, that is $(N \oplus M) \oplus P = N \oplus (M \oplus P)$.

– Show that 0 is identity element, that is $N \oplus 0 = N$.

– Show that any positive integer $N$ has a unique inverse (for this new operation $\oplus$). An element $M$ is an inverse if $N \oplus M = 0$. (Is this a group?)

– Show that for any $N$, adding $(N \oplus \cdots \oplus N)$ itself a finite number of times gives 0. Can you find how many times do you need to add itself for a given $N$.

• Show that $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ is an integer for any choice of positive integers $n$ and $k$ satisfying $k \leq n$. Hint: Use the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for $1 \leq k \leq n - 1$.

Please justify all your answers.