1. For each of the functions below, describe the domain of definition that is understood:
   (a) \[ f(z) = \frac{1}{z^2 + 1}; \]  
   (b) \[ f(z) = \text{Arg} \left( \frac{1}{z} \right); \]  
   (c) \[ f(z) = \frac{z}{z + \overline{z}}; \]  
   (d) \[ f(z) = \frac{1}{1 - |z|^2}. \]  
   Answers: (a) \( z \neq \pm i \); (b) \( \Re z \neq 0 \).

2. In each case, write the function \( f(z) \) in the form \( f(z) = u(x, y) + iv(x, y) \):
   (a) \[ f(z) = z^3 + z + 1; \]  
   (b) \[ f(z) = \frac{\overline{z}^2}{z} \quad (z \neq 0). \]  
   Suggestion: In part (b), start by multiplying the numerator and denominator by \( \overline{z} \).
   Answers: (a) \[ f(z) = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y); \]  
   (b) \[ f(z) = \frac{x^3 - 3xy^2}{x^2 + y^2} \rightleftharpoons \frac{y^3 - 3x^2y}{x^2 + y^2}. \]

3. Suppose that \[ f(z) = x^2 - y^2 - 2y + i(2x - 2xy), \] where \( z = x + iy \). Use the expressions \( x = \frac{z + \overline{z}}{2} \) and \( y = \frac{z - \overline{z}}{2i} \) to write \( f(z) \) in terms of \( z \), and simplify the result.
   Answer: \[ f(z) = z^2 + 2iz. \]

*4. Let \( f(z) = z^2 \). Find the image under \( f \) of the line \( \{ \text{Im } z = 1 \} \). Give an equation for the image curve in terms of \( u \) and \( v \), and sketch the curve in the \( w \)-plane.

5. Exercise 5 of Section 14 in the book.

6. Find and sketch, showing corresponding orientations, the images of the hyperbolas
   \[ x^2 - y^2 = c_1 \quad (c_1 < 0) \]  
   \[ 2xy = c_2 \quad (c_2 < 0) \]
   under the transformation \( w = z^2 \) (i.e. under the function \( f(z) = z^2 \)).
   Comment: See the textbook (Section 14) for the cases where \( c_1 > 0 \) and \( c_2 > 0 \).

7. Show that \( \lim_{z \to 0} \left( \frac{z}{\overline{z}} \right)^2 \) does not exist.
   Hint: Argue much like we did for \( \lim_{z \to 0} \frac{z}{\overline{z}} \). Show that while \( f(z) = \left( \frac{z}{\overline{z}} \right)^2 \) (\( z \neq 0 \)) takes the value 1 on both the real and imaginary axis, the function \( f(z) \) takes the value \(-1\) on the line \( y = x \).

8. Show that
   \[ \lim_{z \to z_0} f(z)g(z) = 0 \quad \text{if} \quad \lim_{z \to z_0} f(z) = 0 \]
   and if there exists a positive number \( M \) such that \( |g(z)| \leq M \) for all \( z \) in some neighborhood of \( z_0 \).
9. Show that
\[ \lim_{z \to \infty} \frac{4z^2}{(z - 1)^2} = 4; \] \[ \lim_{z \to 1} \frac{1}{(z - 1)^3} = \infty; \] \[ \lim_{z \to 1} \frac{z^2 + 1}{z - 1} = \infty. \]

**10. Möbius transformations.** Let \( a, b, c, d \in \mathbb{C} \) with \( ad - bc \neq 0 \). The function
\[ T(z) = \frac{az + b}{cz + d} \quad (z \neq -d/c) \]
is called a Möbius transformation (or linear fractional transformation). Show that
\[ \lim_{z \to \infty} T(z) = \infty \text{ if } c = 0; \]
\[ \lim_{z \to \infty} T(z) = \frac{a}{c} \text{ and } \lim_{z \to -d/c} T(z) = \infty \text{ if } c \neq 0. \]

Here is a very nice [video](#) illustrating and explaining how Möbius transformations work.

**11.** Use the basic rules of differentiation (i.e. \((z^n)’ = nz^{n-1}\), product rule, quotient rule, chain rule, ...) to find \( f'(z) \) when
\[ (a) \quad f(z) = 3z^2 - 2z + 4; \quad (b) \quad f(z) = (2z^2 + i)^5; \]
\[ (c) \quad f(z) = \frac{z - 1}{2z + 1} \quad (z \neq -1/2); \quad (d) \quad f(z) = \frac{(1 + z^2)^4}{z^2} \quad (z \neq 0). \]

12. Show that
\[ \text{a polynomial} \quad P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n \quad (a_0, a_1, \ldots, a_n \in \mathbb{C}, \ a_n \neq 0) \]
or degree \( n \ (n \geq 1) \) is differentiable everywhere, with derivative
\[ P'(z) = a_1 + 2a_2z + \cdots + na_nz^{n-1}; \]
\[ \text{b)} \quad \text{the coefficients of the polynomial } P(z) \text{ in part (a) can be written} \]
\[ a_0 = P(0), \quad a_1 = \frac{P'(0)}{1!}, \quad a_2 = \frac{P''(0)}{2!}, \quad \ldots, \quad a_n = \frac{P^{(n)}(0)}{n!}. \]

13. Suppose that \( f(z_0) = g(z_0) = 0 \) and that \( f'(z_0) \) and \( g'(z_0) \) exist, with \( g'(z_0) \neq 0 \). Use the definition of the derivative to show that
\[ \lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}. \]

**14.** Use the fact that the real and imaginary parts of a complex differentiable function must satisfy the Cauchy-Riemann equations to show that \( f'(z) \) does not exist at any point if
\[ (a) \quad f(z) = \bar{z}; \quad (b) \quad f(z) = z + \bar{z}; \quad (c) \quad f(z) = 2x + ixy^2; \quad (d) \quad f(z) = e^x e^{-iy}. \]

**15.** Determine where \( f'(z) \) exists and find its value when \( f(z) = x^2 + iy^2 \ (z = x + iy) \).

Answer: \( f \) is only complex differentiable when \( x = y \), and \( f'(x + ix) = 2x \).