*1. **Wirtinger derivatives.** (Cf. Section 24, Exercise 8.)

(a) Recall that if \( z = x + iy \), then

\[
x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.
\]

By formally applying the chain rule in calculus to a differentiable function \( F(x, y) \) of two real variables, derive the expression

\[
\frac{\partial F}{\partial z} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial z} = \frac{1}{2} \left( \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y} \right).
\]

(b) Define the operator

\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),
\]

suggested by part (a), to show that if the first-order partial derivatives of the real and imaginary components of a function \( f(z) = u(x, y) + iv(x, y) \) satisfy the Cauchy-Riemann equations, then

\[
\frac{\partial f}{\partial z} = \frac{1}{2} [(u_x - v_y) + i(v_x + u_y)] = 0.
\]

Thus derive the **complex form** \( \frac{\partial f}{\partial z} = 0 \) of the Cauchy-Riemann equations.

(c) By the same kind of formal chain rule argument as in (a) one can also “derive” the formula

\[
\frac{\partial}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).
\]

The operators \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \overline{z}} \) are called **Wirtinger derivatives**. From part (b) we know that a complex function \( f \) is complex differentiable (at a given point) if and only if \( \frac{\partial f}{\partial z} = 0 \) (at that point). Show that if \( f \) is complex differentiable at \( z \), then \( f'(z) = \frac{\partial f}{\partial z} \).

2. Use the Cauchy-Riemann equations to verify that the following functions are entire:

(a) \( f(z) = \cosh x \cos y + i \sinh x \sin y \);

(b) \( f(z) = e^{-y} \sin x - ie^{-y} \cos x \).

*3. Use the Cauchy-Riemann equations to show that each of these functions is nowhere analytic:

(a) \( f(z) = xy + iy \);

(b) \( f(z) = 2xy + i(x^2 - y^2) \).

*4. Determine the singular points of the following functions, and state why the function is analytic everywhere else:

(a) \( f(z) = \frac{2z + 1}{z(z^2 + 1)} \);

(b) \( f(z) = \frac{z^3 + i}{z^2 - 3z + 2} \);

(c) \( f(z) = e^{1/z} \).
5. Let the function \( f(z) = u(r, \theta) + iv(r\theta) \) be analytic in a domain \( D \) that does not include the origin. Using the Cauchy-Riemann equations in polar coordinates and assuming that \( u \) and \( v \) are twice differentiable (which we will show later), show that the function \( u(r, \theta) \) satisfies the partial differential equation
\[
 r^2u_{rr}(r,\theta) + ru_r(r,\theta) + u_{\theta\theta} = 0,
\]
which is the \textit{polar form of Laplace’s equation}. Show that the same is true of the function \( v(r, \theta) \).

6. Let the function \( f(z) = u(x, y) + iv(x, y) \) be analytic in a domain \( D \), and consider the family of \textit{level curves} \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \), where \( c_1 \) and \( c_2 \) are arbitrary real constants. Prove that these families are orthogonal. More precisely, show that if \( z_0 = x_0 + iy_0 \) is a point in \( D \) which lies on two particular curves \( u(x, y) = c_1 \) and \( v(x, y) = c_2 \), and if \( f’(z_0) \neq 0 \), then the lines tangent to those curves at \((x_0, y_0)\) are perpendicular.

Hint: Recall that the gradient vector field of \( u \) (denoted \( \nabla u \) or \( \text{grad} \ u \)) points in the direction of steepest increase of \( u \) and is always perpendicular to the level sets of \( u \) (provided \( \nabla u \neq 0 \)). Hence to show that a level set of \( u \) is perpendicular to a level set of \( v \) at a point of intersection \( z_0 = x_0 + iy_0 \) (provided \( f’(z_0) \neq 0 \)) we just need to show that \( \nabla u \) is perpendicular to \( \nabla v \) (i.e. \( \nabla u \cdot \nabla v = 0 \)). This follows easily from the Cauchy-Riemann equations.

7. Sketch the families of level curves of \( u \) and \( v \) discussed in problem 6 for the functions \( f(z) = z^2 \) and \( f(z) = \frac{1}{z} \).