Math 120A – Homework Assignment 4
Due Thursday, February 7

You only need to turn in solutions to the exercises marked with a asterisk. Please show all of your working.

*1. Find all values of $z$ such that
   
   (a) $e^z = -2$;  
   (b) $e^z = 1 + i$;  
   (c) $e^{2z-1} = 1$.

2. Suppose the function $f(z) = u(x, y) + iv(x, y)$ is analytic on some domain $D$. State why the functions

   $U(x, y) = e^{u(x, y)} \cos v(x, y)$,  
   $V(x, y) = e^{u(x, y)} \sin v(x, y)$

   are harmonic in $D$.

3. Show that

   (a) $\text{Log}(-ei) = 1 - \frac{\pi}{2}i$; 
   (b) $\text{Log}(1 - i) = \frac{1}{2} \ln 2 - \frac{\pi}{4}i$.

*4. Find $\text{Log}(i)$.

5. Show that $\text{Log}(i^3) \neq 3\text{Log}(i)$.

*6. Show that for any two nonzero complex numbers $z_1$ and $z_2$,

   \[
   \text{Log}(z_1 z_2) = \text{Log} z_1 + \text{Log} z_2 + 2N\pi i
   \]

   where $N$ is one of the values 0, ±1.

*7. Show that

   (a) $(1 + i)^n = \exp \left(-\frac{\pi}{4} + 2n\pi\right) \exp \left(i \frac{\ln 2}{2}\right)$, $n \in \mathbb{Z}$;

   (b) $\frac{1}{i^{2n}} = \exp[(4n + 1)\pi i]$, $n \in \mathbb{Z}$.

8. Show that if $z \neq 0$ and $a$ is a real number, then $|z^a| = \exp(a \text{ln} |z|) = |z|^a$, where the principal value of $|z|^a$ is to be taken.

9. Let $c = a + bi$ be a fixed complex number, $c \notin \mathbb{Z}$, and note that $i^c$ is multiple-valued. What additional restriction must be placed on the constant $c$ so that the values of $|i^c|$ are all the same?

   Answer: $c$ must be real.

10. Let $z, c_1, c_2$ be complex numbers, where $z \neq 0$. Prove that if all of the powers involved are principal values, then $z^{c_1+c_2} = z^{c_1} z^{c_2}$.

11. Using the definition of $\sin z$, write down the proof that $(\sin z)' = \cos z$.

12. Prove that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$. Hint: One approach is to start by showing that the entire function $f(z) = \sin^2 z + \cos^2 z$ satisfies $f'(z) = 0$ for all $z \in \mathbb{C}$. 

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13. By writing \( \sin z \) and \( \cos z \) in their real and imaginary parts, show that

\[
|\sin z|^2 = \sin^2 x + \sinh^2 y, \\
|\cos z|^2 = \cos^2 x + \cosh^2 y.
\]

Remark: This shows that \( \sin z \) and \( \cos z \) are unbounded in the complex plane (they get large when \( |y| \) gets large).

14. (a) Using that \( e^{iz} = \cos z + i \sin z \), show that

\[
e^{iz_1}e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).
\]

Then, using that \( \cos(-z) = \cos(z) \) and \( \sin(-z) = -\sin(z) \), use the above to show that

\[
e^{-iz_1}e^{-iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 - i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2).
\]

(b) Use the definition of \( \sin \) to show that

\[
\sin(z_1 + z_2) = \frac{1}{2i} \left[ e^{i(z_1+z_2)} - e^{-i(z_1+z_2)} \right] = \frac{1}{2i} \left( e^{iz_1}e^{iz_2} - e^{-iz_1}e^{-iz_2} \right)
\]

and then use part (a) to obtain the identity

\[
\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.
\]

*15. Use the Cauchy-Riemann equations to show that \( f(z) = \sin(\bar{z}) \) is nowhere analytic.

16. Using the definition of \( \cosh z \), write down the proof that \( (\cosh z)' = \sinh z \).

*17. Show that

(a) \( \sinh(z + \pi i) = -\sinh(z) \);  
(b) \( \cosh(z + \pi i) = -\cosh(z) \).

18. Making use of the fact that \( \sinh(z) = \sinh x \cos y + i \cosh x \sin y \), find all roots of the equation \( \sinh(z) = i \).

Answer: \( z = \left( 2n + \frac{1}{2} \right) \pi i, \quad n \in \mathbb{Z} \).