You only need to turn in solutions to the exercises marked with an asterisk. Please show all of your working.

1. Suppose that \( f(z) \) is entire and the harmonic function \( u(x, y) = \text{Re} \, f(z) \) has an upper bound \( u_0 \); that is, \( u(x, y) \leq u_0 \) for all points \((x, y)\) in the \( xy \)-plane. Show that \( u(x, y) \) must be constant throughout the plane.

   Suggestion: Apply Liouville’s theorem to the function \( g(z) = \exp[f(z)] \).

*2. By substituting \( z = re^{i\theta} \), where \( 0 < r < 1 \), in the formula for the geometric series and taking real and imaginary parts (and then subtracting the \( n = 0 \) term from the sum), show that

\[
\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r \cos \theta - r^2}{1 - 2r \cos \theta + r^2}
\]

and

\[
\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r \sin \theta}{1 - 2r \cos \theta + r^2}
\]

where \( 0 < r < 1 \). (Note that these formulae also hold when \( r = 0 \).)

3. Obtain the Maclaurin series representation

\[
z \cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}, \quad |z| < \infty.
\]

*4. Obtain the Taylor series

\[
e^z = e \sum_{n=0}^{\infty} \frac{(z - 1)^n}{n!}, \quad |z - 1| < \infty
\]

for the function \( f(z) = e^z \) by

(a) using \( f^{(n)}(1) \), for \( n = 0, 1, 2, \ldots \); \hspace{1cm} (b) writing \( e^z = e^{z-1}e \).

5. Rederive the Maclaurin series for \( \cos z \) by using the Maclaurin series for \( e^z \) and the fact that

\[
\cos z = \frac{e^{iz} + e^{-iz}}{2}.
\]

6. With the aid of the identity \( \cos z = -\sin(z - \frac{\pi}{2}) \) expand \( \cos z \) into a Taylor series about the point \( z_0 = \frac{\pi}{2} \).

7. Use the Maclaurin series for \( \sin z \) to write the Maclaurin series for the function \( f(z) = \sin(z^2) \), and point out how it follows that \( f^{(4n)}(0) = 0 \) and \( f^{(2n+1)}(0) = 0 \) for \( n = 0, 1, 2, \ldots \).

8. Derive the Laurent expansion

\[
\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!}, \quad 0 < |z| < \infty.
\]
9. Find the Laurent series that represents the function \( f(z) = z^2 \sin \left( \frac{1}{z} \right) \) in the domain \( 0 < |z| < \infty \).

Answer: \( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}}. \)

10. Find a representation for the function

\[
f(z) = \frac{1}{1 + z} = \frac{1}{z} \cdot \frac{1}{1 + (1/z)}
\]

in negative powers of \( z \) that is valid when \( 1 < |z| < \infty \).

Answer: \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n}. \)

*11. Find the Laurent series that represents the function \( f(z) = \frac{1}{z(1 + z^2)} \) when \( 1 < |z| < \infty \).

Answer: \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^{2n+1}}. \)

Comment: The Laurent series for \( f(z) \) when \( 0 < |z| < 1 \) is found in Example 1 of Section 68 in the book.

*12. By differentiating the Maclaurin series representation \( \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n, |z| < 1, \) obtain the expansions

\[
\frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n + 1)z^n \quad \text{and} \quad \frac{2}{(1 - z)^3} = \sum_{n=0}^{\infty} (n + 1)(n + 2)z^n, \quad |z| < 1.
\]

*13. By substituting \( 1/(1 - z) \) for \( z \) in the expansion \( \frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n + 1)z^n, |z| < 1, \) found in the previous exercise, derive the Laurent series representation

\[
\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(z - 1)^n}, \quad 1 < |z - 1| < \infty.
\]

Comment: The series representation for \( \frac{1}{z^2} \) on \( |z - 1| < 1 \) is given in Example 2 of Section 71.

*14. Show that the function defined by means of the equations

\[
f(z) = \begin{cases} (1 - \cos z)/z^2 & \text{when } z \neq 0, \\ 1/2 & \text{when } z = 0 \end{cases}
\]

is entire. (See Example 1 of Section 71 for a similar problem.)