Math 120A – Homework Assignment 8
Due Friday, March 15

You only need to turn in solutions to the exercises marked with an asterisk. Please show all of your working.

1. Use multiplication of series to show that
\[
\frac{e^z}{z(z^2 + 1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots , \quad 0 < |z| < 1.
\]
You may use that
\[
\frac{1}{z(z^2 + 1)} = \frac{1}{z} + \sum_{n=0}^{\infty} (-1)^{n+1}z^{2n+1} \text{ for } 0 < |z| < 1.
\]

2. By multiplying two Maclaurin series term by term, show that
\[
e^z \sin z = z + z^2 + \frac{1}{3}z^3 + \cdots \text{ for } |z| < \infty.
\]

3. By writing \(\csc z = 1/\sin z\) and then using division, show that
\[
\csc z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^3 + \cdots , \quad 0 < |z| < \pi.
\]
Comment: Recall that the only zeros of \(\sin z\) are \(z = n\pi, n \in \mathbb{Z}\); the function \(\csc z\) is defined away from these points.

4. From the Laurent series for \(\frac{1}{\sinh z}\) derived in Example 2 of Section 73 we have the expansion
\[
\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6} \frac{1}{z} + \frac{7}{360} z + \cdots , \quad 0 < |z| < \pi.
\]
Without using the Residue Theorem, use this expansion to show that
\[
\int_C \frac{dz}{z^2 \sinh z} = -\frac{\pi i}{3}
\]
when \(C\) is the positively oriented unit circle \(|z| = 1\).
Suggestion: Write \(\int_C \frac{dz}{z^2 \sinh z} = \int_C \frac{dz}{z^3} - \frac{1}{6} \int_C \frac{dz}{z} + \int_C \left(\frac{7}{360}z + \cdots\right) dz\) and then explain why the first and third integrals on the RHS are zero.

*5. Find the residue at \(z = 0\) of the function
(a) \(\frac{1}{z + z^2}\); \quad (b) \(z \cos \left(\frac{1}{z}\right)\); \quad (c) \(\frac{\sinh z}{z^4(1 - z^2)}\).

Answers: (a) 1; \quad (b) \(-1/2\); \quad (c) 7/6.

6. Use the Residue Theorem to evaluate the integrals of each of these functions around the circle \(|z| = 3\) in the positive sense:
(a) \(\frac{\exp(-z)}{z^2}\); \quad (b) \(z^2 \exp \left(\frac{1}{z}\right)\).

Answers: (a) \(-2\pi i\); \quad (b) \(\pi i/3\).
7. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

(a) \( z \exp \left( \frac{1}{z} \right) \);  
(b) \( \frac{z^2}{1+z} \);  
(c) \( \frac{\sin z}{z} \);  
(d) \( \frac{\cos z}{z} \);  
(e) \( \frac{1}{(2-z)^3} \).

8. In each case, show that any singular point of the function is a pole. Determine the order \( m \) of each pole, and find the corresponding residue \( B \).

(a) \( \frac{z + 1}{z^2 + 9} \);  
(b) \( \frac{z^2 + 2}{z - 1} \);  
(c) \( \left( \frac{z}{2z + 1} \right)^3 \);  
(d) \( \frac{e^z}{z^2 + \pi^2} \).

Answers: (a) \( m = 1, B = \frac{3 \pm i}{6} \); (b) \( m = 1, B = 3 \); (c) \( m = 3, B = -\frac{3}{16} \); (d) \( m = 1, B = \pm \frac{i}{2\pi} \).

*9. Find the value of the integral \( \int_C \frac{3z^2 + 2}{(z - 1)(z^2 + 9)} \, dz \), taken counterclockwise around the circle (a) \( |z - 2| = 2 \); (b) \( |z| = 4 \).

Answers: (a) \( \pi i \); (b) \( 6\pi i \).

10. Compute the residue at \( z = 0 \) of \( \csc z = \frac{1}{\sin z} \). Can you think of two ways to do this?

11. Use residues to derive the integration formula \( \int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4} \).

*12. Use residues to derive the integration formula \( \int_0^\infty \frac{x^2 \, dx}{x^6 + 1} = \frac{\pi}{6} \).

Comment: Note that \( \int_0^\infty \frac{dx}{x^6 + 1} \) is computed in Section 86 of the textbook.

13. Use residues and the contour shown below to derive the formula \( \int_0^\infty \frac{dx}{x^3 + 1} = \frac{2\pi}{3\sqrt{3}} \).