Math 120B – Homework Assignment 4
Due Thursday, May 9

You only need to turn in solutions to the exercises marked with an asterisk. Please show all of your working.

*1. Show that the transformation \( w = iz + i \) maps the half plane \( x > 0 \) onto the half plane \( v > 1 \).

2. Find a linear transformation that maps the semi-infinite strip \( x > 0, 0 < y < 2 \) onto the strip \( -1 < u < 1, v > 1 \). (Answer: \( w = iz + 1 \)).

*3. Find and sketch the region onto which the half plane \( y > 0 \) is mapped by the transformation \( w = (1 + i)z \).

For problems 4-5 it may be useful to look at Examples 1-3 in Section 98 of the book (Examples 2-3 are also in the lecture notes, at the end of notes 10).

*4. Show that when \( c_1 < 0 \), the image of the half plane \( x < c_1 \) under the transformation \( w = 1/z \) is the interior of a circle. What is the image when \( c_1 = 0 \)?

*5. Show that the image of the half plane \( y > c_2 \) under the transformation \( w = 1/z \) is the interior of a circle when \( c_2 > 0 \). Find the image when \( c_2 < 0 \) and when \( c_2 = 0 \).

6. Show that when a circle is transformed into a circle under the transformation \( w = 1/z \), the center of the original circle is never mapped onto the center of the image circle.

*7. By writing \( z = re^{i\theta} \), show that the transformation \( w = z + \frac{1}{z} \) maps circles \( r = r_0 \) onto ellipses with parametric representations

\[
\begin{align*}
  u &= \left( r_0 + \frac{1}{r_0} \right) \cos \theta, \\
  v &= \left( r_0 - \frac{1}{r_0} \right) \sin \theta \\
  &\quad (0 \leq \theta \leq 2\pi)
\end{align*}
\]

and foci at the points \( w = \pm 2 \). Then show how it follows that this transformation maps the entire circle \( |z| = 1 \) onto the segment \( -2 \leq u \leq 2 \) of the \( u \) axis and the domain outside that circle onto the rest of the \( w \) plane.

*8. Find the linear fractional transformation that maps the points \( z_1 = -i, z_2 = 0, z_3 = i \) onto the points \( w_1 = -1, w_2 = i, w_3 = 1 \), respectively. Into what curve is the imaginary axis \( x = 0 \) transformed?

*9. Find the linear fractional transformation that maps the points \( z_1 = \infty, z_2 = i, z_3 = 0 \) onto the points \( w_1 = 0, w_2 = i, w_3 = \infty \), respectively. (Answer: \( w = -1/z \)).

10. Find the linear fractional transformation that maps the distinct points \( z_1, z_2, z_3 \) onto the points \( w_1 = 0, w_2 = 1, w_3 = \infty \), respectively.

Answer: \( w = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_3)} \). Cf. Section 100, or notes 11.

11. A **fixed point** of a transformation \( w = f(z) \) is a point \( z_0 \) such that \( f(z_0) = z_0 \). Show that every linear fractional transformation, with the exception of the identity transformation \( w = z \), has at most two fixed points in the extended complex plane \( \hat{\mathbb{C}} \).

Comment: This is proved in notes 11.