Math 120B – Homework Assignment 5
Due Thursday, May 16

You only need to turn in solutions to the exercises marked with an asterisk. Note that by the upper half plane we always mean the open upper half plane \((\text{Im } z > 0)\), and by the unit disk we mean the open unit disk \((|z| < 1)\). Please show all of your working.

*1. Observe that if a linear fractional transformation maps the upper half plane one-to-one and onto the upper half plane, then it must also map the real axis (together with \(\infty\)) onto the real axis (together with \(\infty\)). This forces the linear fractional transformation to be of the form

\[
T(z) = \frac{az + b}{cz + d}
\]

where \(a, b, c, d\) are real. (One may first have to cancel a common complex factor in the numerator and the denominator to get the LFT to be of this form.)

(a) Show that if \(a, b, c, d\) are real numbers with \(ad - bc > 0\) then \(T(z)\) maps one-to-one from the upper half plane onto itself.

(b) Show that if \(a, b, c, d\) are real numbers with \(ad - bc < 0\) then \(T(z)\) maps one-to-one from the upper half plane onto the lower half plane.

2. Show that the group of linear fractional transformations mapping one-to-one from the upper half plane onto itself is isomorphic to \(\text{SL}(2, \mathbb{R})\).

*3. The linear fractional transformation \(T(z) = \frac{z - i}{z + i}\) maps one-to one from the upper half plane onto the unit disk. Find the inverse of this map.

*4. Show that any linear fractional transformation mapping one-to-one from the unit disk onto itself is of the form

\[
T(z) = e^{i\beta} \frac{z - a}{1 - \bar{a}z}
\]

where \(\beta \in \mathbb{R}\) and \(a \in \mathbb{C}\) with \(|a| < 1\). One way to do this is to use our classification of the linear fractional transformations mapping one-to-one from the upper half plane onto the unit disk, and the map obtained in problem 4.

5. Show that the transformation \(w = \sin z\) maps one to one from the rectangular region \(0 \leq x \leq \frac{\pi}{2}\), \(0 \leq y \leq 1\) onto the region in the \(w\)-plane described by \(u \geq 0\), \(v \geq 0\),

\[
\frac{u^2}{\cosh^2 1} + \frac{v^2}{\sinh^2 1} \leq 1.
\]
6. Show, indicating the corresponding orientations, that the mapping \( w = z^2 \) transforms horizontal lines \( y = y_1 \quad (y > 0) \) into parabolas \( v^2 = 4y_1^2(u + y_1^2) \), all with foci at the origin \( w = 0 \).

7. Use the result in problem 6 to show that the transformation \( w = z^2 \) maps one-to-one from any horizontal strip \( a \leq y \leq b \) above the \( x \)-axis onto the closed region between the two parabolas \( v^2 = 4a^2(u + a^2) \), \( v^2 = 4b^2(u + b^2) \).

8. Let \( F_0(z) \) denote the principal branch of \( z^{1/2} \). Show that \( w = F_0(z) \) maps one-to-one from the cut plane \( \mathbb{C} \setminus (-\infty, 0] \) onto the half plane \( \text{Re} \ w > 0 \).

9. The branch \( F(z) \) of \( (z^2 - 1)^{1/2} \) discussed in class may be written in terms of the coordinates \( r_1 = |z - 1|, \ \theta_1 = \arg(z - 1), \ r_2 = |z + 1|, \ \theta_2 = \arg(z + 1) \) where \( 0 < \arg z \leq 2\pi \) (see Example 2 of Section 109 in the book, or notes 13). Explain geometrically why the conditions \( r_1 > 0, \ \ 0 < \theta_1 + \theta_2 < \pi \) describe the first quadrant \( x > 0, \ y > 0 \) of the \( z \)-plane. Then show that \( w = F(z) \) maps that quadrant onto the first quadrant \( u > 0, \ v > 0 \) of the \( w \) plane.

\textit{Suggestion:} To show that the quadrant \( x > 0, \ y > 0 \) in the \( z \)-plane is described, note that \( \theta_1 + \theta_2 = \pi \) at each point on the positive \( y \)-axis and that \( \theta_1 + \theta_2 \) decreases as a point \( z \) moves to the right along a ray \( \theta_2 = c \quad (0 < c < \pi/2) \).