1. Let $n$ be a natural number. Prove that 

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$ 

2. Let $m$ and $n$ be natural numbers.

(a) Prove that the sum, $m + n$, is also a natural number. (*Hint:* Fix $m$ and define $S(n)$ to be the statement that $m + n$ is also a natural number.)

(b) Prove that the product, $mn$, also is a natural number. (*Hint:* Fix $m$ and define $S(n)$ to be the statement that $mn$ is also a natural number.)

3. (a) Prove that the sum of a rational number and an irrational number must be irrational.

(b) Prove that the product of two nonzero numbers, one rational and one irrational, is irrational.

4. Let $c$ be a positive number. Prove that there is a positive number whose square is $c$ by completing the following steps: Define 

$$S = \{ x \mid x \in \mathbb{R}, x \geq 0, x^2 < c \}.$$ 

(a) Show that $c + 1$ is an upper bound for $S$ and therefore, by the Completeness Axiom, $S$ has a least upper bound that we denote by $b$.

(b) Show that if $b^2 > c$, then we can choose a suitably small positive number $\epsilon$ such that $b - \epsilon$ is also an upper bound for $S$, thus contradicting the choice of $b$ as the least upper bound of $S$.

(c) Show that if $b^2 < c$, then we can choose a suitably small positive number $\epsilon$ such that $b + \epsilon$ belongs to $S$, thus contradicting the choice of $b$ as an upper bound of $S$.

(d) Use parts (b) and (c) and the Positivity Axioms for $\mathbb{R}$ to conclude that $b^2 = c$.

5. Suppose that $S$ is a nonempty set of integers that is bounded below. Show that $S$ has a minimum. In particular, conclude that every nonempty set of natural numbers has a minimum.

6. Let $S$ be a nonempty set of real numbers that is bounded below. Prove that the set $S$ has a minimum if and only if the number $\inf S$ belongs to $S$.

7. Let $a$ be a real number with the property that for every natural number $n$, $a \leq \frac{1}{n}$. Prove that $a \leq 0$.

8. Given a real number $a$, define $S = \{ x \mid x \in \mathbb{Q}, x < a \}$. Prove that $a = \sup S$.

9. *Cauchy’s Inequality:* Using the fact that the square of a real number is nonnegative, prove that for any real numbers $a$ and $b$,

$$ab \leq \frac{1}{2}(a^2 + b^2).$$

10. *Bernoulli’s Inequality:* Show that for a natural number $n$, and a nonnegative real number $b$,

$$(1 + b)^n \geq 1 + nb.$$