Math 142A : Midterm 2
Introduction to analysis

You have 50 minutes. No books and notes are allowed.
1. True/False: Circle the correct answer. No justifications are needed in this exercise. (1 point each)

(1) Every function \( f : \mathbb{N} \rightarrow \mathbb{R} \) is continuous. \quad \text{T / F}

(2) There is a continuous function \( f : [0, 1] \rightarrow \mathbb{R} \) such that \( f([0, 1]) = (0, 1) \). \quad \text{T / F}

(3) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a function such that \( f(0) = 0 \) and \( f(1) = 2 \). Then there exists \( x \in (0, 1) \) such that \( f(x) = 1 \). \quad \text{T / F}

(4) Let \( f : [0, 1] \rightarrow \mathbb{R} \) be a continuous function that is increasing. Suppose that \( f(0) = 0 \) and \( f(1) = 1 \). Then \( f([0, 1]) = [0, 1] \). \quad \text{T / F}

(5) The function \( f : (0, 1) \rightarrow \mathbb{R} \) given by \( f(x) = x^2 \) is uniformly continuous. \quad \text{T / F}

(6) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a function. Suppose that for all \( \varepsilon > 0 \) we can find \( \delta > 0 \) such that \( |f(x) - f(0)| \leq \varepsilon \), for all \( x \in [-\delta, \delta] \). Then \( f \) is continuous at 0. \quad \text{T / F}

(7) Every decreasing function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is one-to-one. \quad \text{T / F}

(8) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a monotone function such that \( f(\mathbb{R}) = \mathbb{R} \). Then \( f \) is continuous. \quad \text{T / F}

(9) There is a monotone function \( f : [0, 1] \rightarrow \mathbb{R} \) such that \( f([0, 1]) = (0, 1) \). \quad \text{T / F}

(10) Every number \( x_0 \in [0, 1] \) is a limit point of \( \mathbb{Q} \cap (0, 1) \). \quad \text{T / F}
2. Consider a function $f : (0, +\infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } 0 < x < 1, \\
\sqrt{x}, & \text{if } x \geq 1. \end{cases}$$

(a) (5 points) Use the $\varepsilon$-$\delta$ criterion to prove that $f$ is continuous at $x_0 = 1$.
(b) (5 points) Prove that $f$ is not uniformly continuous.
3. (a) (5 points) Prove that there exists $x \in (1, 2)$ such that $x^2 = \sqrt{x^3 + 1}$.

(b) (5 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = 1$, $f(1) = 2$ and $f(2) = 0$. Prove that there exists $x \in \mathbb{R}$ such that $f(x) = f(x + 1)$. 

4. (a) (2 points) State the definition of a limit point of a set $D$.
(b) (4 points) Prove that every real number is a limit point of $\mathbb{R} \setminus \mathbb{Q} = \{ x \in \mathbb{R} | x \notin \mathbb{Q} \}$.
(c) (4 points) Prove that the set $\mathbb{N}$ of natural numbers has no limit points.
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