You only need to turn in problems marked with an asterisk (*).

*1. Let \( f : (0, 2) \to \mathbb{R} \) be given by \( f(x) = 1/x \).

(a) Find \( p_n \), the \( n^{th} \) order Taylor polynomial centered at \( x_0 = 1 \).

(b) Use the Geometric Sum Formula to show that for every natural number \( n \),

\[
f(x) - p_n(x) = \frac{(1 - x)^{n+1}}{x} \quad \text{if } 0 < x < 2.
\]

(c) Use part (b) to prove that

\[
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!} (x-1)^k \quad \text{if } |x-1| < 1.
\]

*2. Suppose that the function \( f : \mathbb{R} \to \mathbb{R} \) has derivatives of all orders and that

\[
\begin{cases}
    f'(x) - f(x) = 0 & \text{for all } x \in \mathbb{R}, \\
    f(0) = 2.
\end{cases}
\]

Find a recursive formula for the coefficients of the \( n^{th} \) Taylor polynomial for \( f \) centered at 0. Use Theorem 8.14 (with \( r > 0 \) arbitrary) to show that the Taylor expansion converges at every point (i.e. the Taylor series centered at 0 converges to the function \( f \) at every point).

3. Suppose that the function \( f : \mathbb{R} \to \mathbb{R} \) has derivatives of all orders and that

\[
\begin{cases}
    f''(x) - f'(x) - f(x) = 0 & \text{for all } x \in \mathbb{R}, \\
    f(0) = 1 \quad \text{and} \quad f'(0) = 1.
\end{cases}
\]

Find a recursive formula for the coefficients of the \( n^{th} \) Taylor polynomial for \( f \) at \( x = 0 \). Show that the Taylor expansion converges at every point (i.e. the Taylor series centered at 0 converges to the function \( f \) at every point).

*4. Let \( \alpha, \beta, A \) and \( B \) be real numbers. Suppose that the function \( f : \mathbb{R} \to \mathbb{R} \) has derivatives of all orders and that

\[
\begin{cases}
    f''(x) + \alpha f'(x) + \beta f(x) = 0 & \text{for all } x \in \mathbb{R}, \\
    f(0) = A \quad \text{and} \quad f'(0) = B.
\end{cases}
\]

Find a recursive formula for the coefficients of the \( n^{th} \) Taylor polynomial for \( f \) at \( x = 0 \). Show that the Taylor expansion converges at every point (i.e. the Taylor series centered at 0 converges to the function \( f \) at every point).
5. In class we obtained the following formula

\[ \ln(1 + x) = p_n(x) + \int_1^{1+x} \frac{(1 - t)^n}{t} \, dt, \]

for any \( n \in \mathbb{N} \) and \( x \in (-1, \infty) \), where \( p_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}x^k}{k} \). Using this one can show (see the proof of Theorem 8.16 in the book or in the notes on Section 8.4) that for each natural number \( n \),

\[ \left| \ln(1 + x) - \sum_{k=1}^{n} \frac{(-1)^{k+1}x^k}{k} \right| \leq \frac{1}{1 + x} \frac{|x|^{n+1}}{n+1} \quad \text{if } -1 < x \leq 0, \]

and

\[ \left| \ln(1 + x) - \sum_{k=1}^{n} \frac{(-1)^{k+1}x^k}{k} \right| \leq \frac{x^{n+1}}{n+1} \quad \text{if } 0 \leq x \leq 1. \]

Use the latter inequality to estimate \( \ln(1.1) \) with an error of at most \( 10^{-4} \).