A BRIEF INTRODUCTION TO LOCAL COHOMOLOGY

SAMIR CANNING

ABSTRACT. This paper is a final paper for A.J de Jong's Spring 2017 course in Algebraic Geometry at Columbia University. It concerns the local cohomology of a module over a Noetherian ring with support in an ideal $\mathfrak{a}$. The goal of the paper is to introduce local cohomology, and relate it to sheaf cohomology via Čech cohomology.

1. DEFINITIONS

Throughout this section, we fix a commutative Noetherian ring R. We denote the category of modules over R as Mod_R.

Definition 1.1. Let $\mathfrak{a} \subset R$ be an ideal. The $\mathfrak{a}$-torsion functor, $\Gamma_{\mathfrak{a}} : \mathrm{Mod}_R \to \mathrm{Mod}_R$, is defined as follows:
For $M \in \mathrm{Mod}_R$

$$\Gamma_{\mathfrak{a}}(M) = \{m \in M : \mathfrak{a}^t m = 0 \text{ for some } t > 0\}$$

and for $\varphi : M \to N$ a homomorphism of R-modules,

$$\Gamma_{\mathfrak{a}}(\varphi) = \varphi|_{\Gamma_{\mathfrak{a}}(M)} : \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(N)$$

The following easy lemma allows us to define the right derived functors of the $\mathfrak{a}$-torsion functor.

Lemma 1.2. *The $\mathfrak{a}$-torsion functor is left exact.*

Definition 1.3. The i^{th} local cohomology functor with support in the ideal $\mathfrak{a}$, denoted $H^i_{\mathfrak{a}}(-)$ is the i^{th} right derived functor of $\Gamma_{\mathfrak{a}}(-)$. That is,

$$H^i_{\mathfrak{a}}(M) = H^i_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(I^\bullet))$$

where $M \to I^\bullet$ is an injective resolution of $M \in \mathrm{Mod}_R$.

2. ALTERNATIVE CHARACTERIZATIONS OF LOCAL COHOMOLOGY

We now give several alternate characterizations of local cohomology. These characterizations will allow us to relate it to Čech cohomology, and thus to sheaf cohomology. Our first characterization is the following theorem.

Theorem 2.1. *Let R be a commutative Noetherian ring and M an R-module. There are natural isomorphisms*

$$\underset{t}{\underrightarrow{\mathrm{colim}}}\, \mathrm{Ext}^i_R(R/\mathfrak{a}^t, M) \cong H^i_{\mathfrak{a}}(M)$$

Proof. First, we note that for any R-module N, we have for each t the isomorphism

$$\mathrm{Hom}_{\mathrm{R}}(R/\mathfrak{a}^t, N) \cong \{n \in N : \mathfrak{a}^t n = 0\}$$
$$\varphi \mapsto \varphi(1)$$

It is clear that if we take the colimit over t on the right hand side, we get $\Gamma_{\mathfrak{a}}(N)$.

Hence,

$$\varinjlim_t \mathrm{Hom_R}(R/\mathfrak{a}^t, N) \cong \Gamma_{\mathfrak{a}}(N)$$

This extends to an isomorphism of complexes because the above construction is functorial. Let $I^\bullet$ be an injective resolution of M. Because cohomology commutes with colimits, we get

$$\varinjlim_t H^i(\mathrm{Hom_R}(R/\mathfrak{a}^t, I^\bullet) \cong \varinjlim_t \mathrm{Ext}^i_R(R/\mathfrak{a}^t, M) \cong H^i_{\mathfrak{a}}(M)$$

□

For our next characterizations, we begin by review the definitions of the Čech complex and the Koszul complex.

Definition 2.2. Let $f \in R$, a commutative but not necessarily Noetherian ring. The Čech complex on f is

$$0 \to R \to R_f \to 0$$

where the middle map is localization, and R and R_f are in degrees 0 and 1, respectively. We denote it by $\check{C}^\bullet(f;R)$. Let $\mathbf{f} = f_1, \cdots, f_s$ be a sequence of elements in R. Then the Čech complex on $\mathbf{f}$ is

$$\check{C}^\bullet(\mathbf{f};R) = \check{C}^\bullet(f_1;R) \otimes_R \cdots \otimes_R \check{C}^\bullet(f_s;R)$$

If M is an R-module, then $\check{C}^\bullet(\mathbf{f};M) = \check{C}^\bullet(\mathbf{f};R) \otimes_R M$ The Čech cohomology is the cohomology of the Čech complex:

$$\check{H}^i(\mathbf{f};M) = H^i(\check{C}^\bullet(\mathbf{f};M))$$

Definition 2.3. Let $f \in R$, a commutative but not necessarily Noetherian ring. The Koszul complex on f is

$$0 \to R \to R \to 0$$

where the middle map is multiplication by f, and the R's are in degrees -1 and 0. We denote it by $K^\bullet(f;R)$. Let $\mathbf{f} = f_1, \cdots, f_s$ be a sequence of elements in R. Then the Koszul complex on $\mathbf{f}$ is

$$K^\bullet(\mathbf{f};R) = K^\bullet(f_1;R) \otimes_R \cdots \otimes_R K^\bullet(f_s;R)$$

If M is an R-module, then $K^\bullet(\mathbf{f};M) = K^\bullet(\mathbf{f};R) \otimes_R M$. The Koszul cohomology is the cohomology of the Koszul complex:

$$H^i_K(\mathbf{f};M) = H^i(K^\bullet(\mathbf{f};M))$$

Notationally, we let $\mathbf{f}^t = f_1^t, \cdots, f_s^t$ where $\mathbf{f} = f_1, \cdots f_s$ is a sequence of elements in R.

Lemma 2.4. *The Koszul complex is self dual. That is, if $\mathbf{f} = f_1, \cdots, f_s$ are elements of R and M is an R-module, then*

$$K^\bullet(\mathbf{f};M) \cong \mathrm{Hom_R}(K^\bullet(\mathbf{f};R), M)[s]$$

Proof. See [2] page 69. □

Lemma 2.5. *Let $\mathbf{f} = f_1, \cdots, f_s$ be a sequence of elements in a ring R. Let M be an R-module. Then*

$$\check{C}^\bullet(\mathbf{f}, M) = \varinjlim_t (K^\bullet(\mathbf{f}^t;R) \otimes_R M)[-s]$$

Proof. Denote the complex on the right hand side of the equality by $C^\bullet(\mathbf{f}, M)$. It suffices to consider the case where $\mathbf{f} = f$ is a single element in R because of the tensor product decomposition of the Koszul complex. Further, it suffices to consider the case $M = R$ because taking colimits commutes with taking tensor products. The directed system we're considering is the system of complexes that looks like

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{f^t} & R & \longrightarrow & 0 \\ & & {\scriptstyle 1}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \\ 0 & \longrightarrow & R & \xrightarrow{f^{t+1}} & R & \longrightarrow & 0 \end{array}$$

where the vertical maps are the structure maps. Because the colimit of the system

$$R \xrightarrow{f} R \xrightarrow{f} R \xrightarrow{f} \cdots$$

is R_f, the colimit of our system of complexes is

$$0 \longrightarrow R \longrightarrow R_f \longrightarrow 0$$

Here, R is in degree 0. This complex is precisely how we defined the Čech complex, so we get the desired equivalence. □

Theorem 2.6. *Let $\mathbf{f} = f_1, \cdots, f_s$ generate the ideal $\mathfrak{a}$ of the commutative Noetherian ring R. There is an isomorphism of functors $Mod_R \to Mod_R$*

$$\theta : H^\bullet_{\mathfrak{a}}(-) \xrightarrow{\cong} \varinjlim_t (\mathrm{Hom_R}(K^\bullet(\mathbf{f}^t; R), -))$$

Proof. We first construct a homomorphism $\theta^i_t : Ext^j(R/\mathbf{f}^t R, M) \to H^i(\mathrm{Hom_R}(K^\bullet(\mathbf{f}^t; R), M))$ for each i and each t. We make it so that this homomorphism is compatible with the directed systems, so that we get commutative diagrams of the form:

$$\begin{array}{ccc} \mathrm{Ext}^i_R(R/\mathbf{f}^t R, M) & \xrightarrow{\theta^i_t} & H^i(\mathrm{Hom_R}(K^\bullet(\mathbf{f}^t; R), M) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Ext}^i_R(R/\mathbf{f}^{t+1} R, M) & \xrightarrow{\theta^i_{t+1}} & H^i(\mathrm{Hom_R}(K^\bullet(\mathbf{f}^{t+1}; R), M)) \end{array}$$

To get the θ maps, we tensor componentwise the directed system of Koszul complexes given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{f^{t+1}} & R & \longrightarrow & 0 \\ & & {\scriptstyle f}\big\downarrow & & {\scriptstyle 1}\big\downarrow & & \\ 0 & \longrightarrow & R & \xrightarrow{f^{t}} & R & \longrightarrow & 0 \end{array}$$

Through tensoring, we get a system of complexes

$$\cdots \longrightarrow K^\bullet(\mathbf{f}^{t+1}; R) \longrightarrow K^\bullet(\mathbf{f}^t; R) \longrightarrow \cdots \longrightarrow K^\bullet(\mathbf{f}; R)$$

Let ϵ_t denote the morphism of complexes $K^\bullet(\mathbf{f}^t; R) \to R/\mathbf{f}^t R$, where the latter is a commplex in degree 0. Let $M \to I^\bullet$ be an injective resolution of M. Then we get morphisms:

$$\mathrm{Hom_R}(R/\mathbf{f}^t R, I^\bullet) \to \mathrm{Hom_R}(K^\bullet(\mathbf{f}^t; R), I^\bullet) \leftarrow \mathrm{Hom_R}(K^\bullet(\mathbf{f}^t; R, M)$$

The morphism on the right is a quasi-isomorphism because the Koszul complex is a bounded complex of free R-modules. Hence, we get our maps θ_t^i that are compatible with our directed system. Thus, we have a map

$$\theta^i(M) : \varinjlim_t \operatorname{Ext}^i_R(R/\mathbf{f}^t R, M) \to \varinjlim_t H^i(\operatorname{Hom_R}(K^\bullet(\mathbf{f}^t; R), M))$$

To see that this is a bijection, we follow a typical procedure for showing two δ-functors are equivalent. That is, we show the 0^{th} map is an isomorphism, show that both sides vanish on injective objects for $i \geq 1$ and then use induction to verify that each map in the natural transformation is an isomorphism. We leave these details out, and refer the reader to [1] pages 30-35. After showing this isomorphism, we simply apply the characterization in theorem 2.1. $\square$

Combining 2.4, 2.5 and 2.6, we get the following theorem.

Theorem 2.7. *Let $\mathbf{f} = f_1, \cdots, f_s$ generate the ideal $\mathfrak{a}$ of R. Then there is an isomorphism of functors*

$$H^\bullet_{\mathfrak{a}}(-) \cong \check{H}^\bullet(\mathbf{f}; -)$$

3. Comparison with Sheaf Cohomology

Theorem 3.1. *Let R be a commutative Noetherian ring, $\mathfrak{a} = (f_1, \cdots, f_s)$ an ideal of R, M a module over R, $\widetilde{M}$ the sheaf associated to M, and $U = \operatorname{Spec}(R) - V(\mathfrak{a})$. Then there is an exact sequence*

$$0 \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow \Gamma(U, \widetilde{M}) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

and for all $i \geq 1$, we have $H^i(U, \widetilde{M}) \cong H^{i+1}_{\mathfrak{a}}(M)$

Proof. Let $U_i = \operatorname{Spec}(R) - V(f_i)$. Then we have that for each $j \geq 1$,

$$U_{i_1} \cap \cdots \cap U_{i_j+1} = \operatorname{Spec}(R_{f_{i_1} \cdots f_{i_j+1}})$$

The intersection is affine, so the restriction of $\widetilde{M}$ to it is quasicoherent. Thus we get vanishing of cohomology for $i \geq 1$ by the Grothendieck vanishing theorem: $H^i(U_{i_1} \cap \cdots \cap U_{i_{j+1}}, \widetilde{M}) = 0$. Because the U_i cover $\operatorname{Spec}(R)$, we get the usual isomorphism of sheaf cohomology and Čech cohomology, $H^i(U, \widetilde{M}) \cong \check{H}^i(\cup_i U_i, \widetilde{M})$. Then, using 2.7, we can compute local cohomology via the Čech complex, giving us exactly what we want. $\square$

The upshot of this last theorem is that the sheaf cohomology of $\widetilde{M}$ away from $V(\mathfrak{a})$ tells us what the local cohomology of M with support in $\mathfrak{a}$ is. Thus, only the support of $\mathfrak{a}$ counts when we compute local cohomology, hence the name local cohomology with support in $\mathfrak{a}$.

References

[1] Mel Hochster *Local Cohomology* Online Lecture Notes: http://www.math.lsa.umich.edu/ hochster/615W11/loc.pdf

[2] Srikanth B. Iyengar et. al. *Twenty-Four Hours of Local Cohomology* American Mathematical Society, Providence, Rhode Island, 2007

Department of Mathematics, Columbia University
E-mail address: src2165@columbia.edu