(1) Let \( n \geq 3 \) be an integer. Let \( K_n \) be the complete graph with vertices 1, \ldots, \( n \) and let \( G = K_n \setminus \{n, n-1\} \) and \( H = K_n/\{n, n-1\} \). Compute the number of spanning trees of \( G \) and also of \( H \).

(2) (a) Given a graph with weights on its edges, describe an algorithm to find a maximum weight spanning tree, i.e., a spanning tree whose weight is as big as possible. Also prove that your method is correct.

(b) Find a minimum weight spanning tree in the following graph. Also, use your algorithm from (a) to find a maximum weight spanning tree. What are their weights?

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (3,1) -- (0,0);
\draw (0,0) -- (1,0) -- (2,1) -- (3,0) -- (0,0);
\draw (1,1) -- (2,1) -- (3,1) -- (1,1);
\draw (2,0) -- (3,0) -- (2,1) -- (2,0);
\draw (1,0) -- (2,0) -- (1,1) -- (1,0);
\draw (0,0) -- (1,0) -- (0,0);
\end{tikzpicture}
\end{center}

(3) What is the chromatic number of the following graph (with 10 vertices)? Explain.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (3,1) -- (0,0);
\draw (0,0) -- (1,0) -- (2,1) -- (3,0) -- (0,0);
\draw (1,1) -- (2,1) -- (3,1) -- (1,1);
\draw (2,0) -- (3,0) -- (2,1) -- (2,0);
\draw (1,0) -- (2,0) -- (1,1) -- (1,0);
\draw (0,0) -- (1,0) -- (0,0);
\end{tikzpicture}
\end{center}

(4) What is the chromatic polynomial of the following graph? Show work or explain.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (3,1) -- (0,0);
\draw (0,0) -- (1,0) -- (2,1) -- (3,0) -- (0,0);
\draw (1,1) -- (2,1) -- (3,1) -- (1,1);
\draw (2,0) -- (3,0) -- (2,1) -- (2,0);
\draw (1,0) -- (2,0) -- (1,1) -- (1,0);
\draw (0,0) -- (1,0) -- (0,0);
\end{tikzpicture}
\end{center}

(5) Let \( G \) be a graph without loops. Let \( n \) be the number of vertices and let \( \chi_G(z) \) be its chromatic polynomial. Recall from HW5 that \( \Omega(G) \) is the number of cycle-free orientations of \( G \). Show that

\[ \Omega(G) = (-1)^n \chi_G(-1). \]
Hints

2a: Your proof should be fairly short and use Kruskal’s greedy algorithm.

5: Do induction on the number of edges and use that both $\Omega(G)$ and $(-1)^n\chi_G(-1)$ satisfy similar kinds of deletion-contraction recursions.