Math 184, Fall 2023
Homework 1
Due: Wednesday, Oct. 11 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences. **I'll put some hints and answers to check your work on the next page.**

(1) (a) How many ways are there to list the letters of ORANGE ?
   (b) How many ways are there to list the letters of BLACKBERRY ?
   (c) How many ways are there to list the letters of ORANGE so that the first letter is *not* a vowel (for us, vowel means AEIOU)?
   (d) How many ways are there to list the letters of BLACKBERRY so that the first letter is *not* a vowel? Make sure to check your work with the next page.

(2) Let $n \geq 2$ be an integer and define $[n] = \{1, 2, \ldots, n\}$. Define the following sets:

$$A = \{S \subseteq [n] \mid 1 \in S \text{ and } 2 \in S\},$$
$$B = \{S \subseteq [n] \mid 1 \in S \text{ or } 2 \in S\}.$$

In words: A is the collection of subsets that contain *both* 1 and 2, while B is the collection of subsets that contain *at least one* of 1 and 2.

Find formulas for the sizes of A and B (with explanation). You can find the numerical answer for $n = 6$ on the next page.

(3) Let n be a positive integer. Let A_n be the set of sequences whose entries are all 1 or 2 such that the sum of the entries is n. For example,

$$A_4 = \{(1,1,1,1), (1,1,2), (1,2,1), (2,1,1), (2,2)\}$$

and $|A_4| = 5$. If $n \geq 3$, prove that

$$|A_n| = |A_{n-1}| + |A_{n-2}|.$$

(4) (a) A classroom has 90 seats and 75 students. How many different seating arrangements are there? (Students cannot share a seat!)
   (b) A neighborhood has 8 houses and each needs to be painted one of three colors: grey, brown, or yellow. How many ways can we choose colors subject to the constraint that there has to be at least two grey houses? Make sure to check your work with the next page.

(5) Let n be a positive integer.
   (a) How many ways can we choose a pair of subsets (S, T) of $[n]$ such that $S \cap T = \emptyset$?
   (b) How many ways can we choose a pair of subsets (S, T) of $[n]$ such that $S \cap T \neq \emptyset$?
   (c) How many ways can we choose a triple of subsets (S, T, U) of $[n]$ such that $S \cup T \cup U = [n]$?
   (d) How many ways can we choose a triple of subsets (S, T, U) of $[n]$ such that $S \cup T \cup U = [n]$ and $S \cap T \cap U \neq \emptyset$?

   Check your work with the next page, numerical answers are given for $n = 6$.

Checking your work

Here I'm going to give you some points of data that you can use to check if your final answer is correct. Usually, the form of your answer will look like a complicated expression and it's fine to leave it like that. These are provided to help make sure you can solve the problems *correctly* without spoiling the solution, and I encourage the use of computers to check your work.

(1): For part (d), your answer should simplify to 725760. Use a calculator or computer to check if you're not sure (don't simplify it by hand!).

(2): You are supposed to get formulas that involve n. If you plug in $n = 6$, you should get $|A| = 16$ and $|B| = 48$. If not, check your work again.

(4): For part (b), your answer should simplify to 5281. Use a calculator or computer to check if you're not sure (don't simplify it by hand!).

(5): You are supposed to get formulas that involve n. If you plug in $n = 6$, you should get these answers: (a) 729 (b) 3367 (c) 117649 (d) 70993. If not, check your work again.

Hints

You will generally get more out of the homework if you try to think about how to solve the problems by yourself without any help. However, it's much better to solve the problem with help than to not solve it at all. So if you get stuck, try reading below. Beyond that, you can ask other students or ask the professor/TAs in discussion, office hours, or Discord.

(2): Adapt the idea in Example 1.1.3.

(3): The formula strongly suggests that you use the sum principle. Try splitting up the sequences in A_n based on what their last entry is.

(5): Adapt the idea in Example 2.2.4. The subtraction principle will be needed for some of these.