Explanations should be given for your solutions. Use complete sentences.

(1) Prove that every polynomial in $x$ can be written as a linear combination of the polynomials

$$1, 2x - 1, (2x - 1)^2, (2x - 1)^3, (2x - 1)^4, \ldots.$$ 

(2) How many ways are there to list the letters of the word MATHEMATICIAN?

(3) How many integers are there between 10000 and 99999 in which all digits are different?

(4) Let $n \geq 3$ be an integer. Define the following sets:

$$A = \{S \subseteq [n] \mid 1 \in S \text{ and } 3 \in S\},$$
$$B = \{S \subseteq [n] \mid 1 \in S \text{ and } 3 \notin S\},$$
$$C = \{S \subseteq [n] \mid 1 \notin S \text{ and } 3 \notin S\},$$
$$D = \{S \subseteq [n] \mid \{1, 3\} \cap S \geq 1\}.$$ 

Find formulas for the size of each set.

(5) (a) We want to select three subsets $A$, $B$, and $C$ of $[n]$ so that $A \subseteq C$ and $B \subseteq C$. How many ways can this be done?

(b) We want to select three subsets $A$, $B$, and $C$ of $[n]$ so that $A \subseteq C$, $B \subseteq C$, and $A \cap B \neq \emptyset$. How many ways can this be done?

(6) Fix a positive integer $n \geq 1$. Let $A_1$ be the set of subsets $S \subseteq [n]$ with no consecutive elements, i.e., if $i \in S$, then $i + 1 \notin S$.

For example, when $n = 3$, $|A_1| = 5$ and $A_1$ is the following set of subsets:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}.$$ 

Let $A_2$ be the set of ways of tiling the $2 \times (n + 1)$ rectangle with the shapes: $2 \times 1$ rectangle and $1 \times 2$ rectangle without any overlaps.

For example, when $n = 3$, $|A_2| = 5$ and $A_2$ is the following set of tilings:

Construct a bijection between $A_1$ and $A_2$ (and prove that it is a bijection). You may use the fact, without proving it, that the following configuration never appears in a tiling:

Hint: Consider the column indices where there are horizontal tiles.