Math 184, Fall 2023
Homework 2
Due: Wednesday, Oct. 18 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences.
I'll put some hints and answers to check your work on the next page.

(1) We consider some variations of standard Poker hands. Start with a standard deck of cards (4 suits, 13 values, so 52 cards in total).
  (a) How many ways can we choose 8 cards so that we have 4 pairs? (A pair is two cards with the same value.) We require that all 4 values are distinct.
  (b) How many ways can we choose 6 cards so that at most 2 suits appear? (i.e., either all cards have the same suit or at worst, there are 2 different suits so that each card has one or the other).

(2) Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 120.$$

For each of the following conditions, how many solutions are there? (Each part is an independent problem, don't combine the conditions.)
  (a) The x_i are all positive even integers.
  (b) The x_i are all positive odd integers.
  (c) x_1, x_2 are positive even integers and x_3, x_4, x_5, x_6 are positive odd integers.

(3) Let n and k be positive integers such that $n \geq 2k - 1$.
Let A_n be the collection of k-element subsets of $[n - k + 1]$.
Let B_n be the collection of k-element subsets of $[n]$ that have no consecutive elements (if $i \in S$ then $i + 1 \notin S$).
Describe a bijection between A_n and B_n. Use this to derive a formula for $|B_n|$.

(4) Let n be a positive integer. Find simple formulas for $S(n + 2, n)$ and $S(n + 3, n)$.

(5) Let Y_n be the collection of set partitions of $[n]$ such that every block has size 2 or 3 and let $y_n = |Y_n|$. If $n \geq 4$, prove that

$$y_n = (n - 1)y_{n-2} + \binom{n-1}{2} y_{n-3}.$$

Checking your work

Here I'm going to give you some points of data that you can use to check if your final answer is correct. Usually, the form of your answer will look like a complicated expression and it's fine to leave it like that. These are provided to help make sure you can solve the problems *correctly* without spoiling the solution, and I encourage the use of computers to check your work.

(1) (a): 926640 (b): 1367652

(2) (a): 5006386 (b): 6471002 (c): 5949147

(4) You should have a formula in terms of n using familiar notation. When $n = 6$, your formulas should have these values: $S(8, 6) = 266$ and $S(9, 6) = 2646$

Hints

You will generally get more out of the homework if you try to think about how to solve the problems by yourself without any help. However, it's much better to solve the problem with help than to not solve it at all. So if you get stuck, try reading below. Beyond that, you can ask other students or ask the professor/TAs in discussion, office hours, or Discord.

(2) (a): Do a change of variables $x_i = 2y_i$. (b): Do a change of variables $x_i = 2y_i - 1$.

(3) Starting from a subset $S \in A_n$, what can you do to its elements to force the non-consecutive condition?

Further hint: Adapt the idea from either Example 2.3.4 or the proof of Theorem 2.3.5.

(4) See Example 3.1.6 for $S(n, n-1)$, which is a simpler version of this problem.

(5) This is another type I/II kind of problem. Think about how big the block containing n is; it's either size 2 or size 3, and those will be your two types.

When building a bijection to a smaller instance of the problem, I recommend deleting the entire block containing n (but remembering what numbers shared the block with it).