Math 184, Fall 2023
Homework 3
Due: Wednesday, Nov. 1 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences.
I'll put some hints and answers to check your work on the next page.

(1) Write x^5 and x^6 as a linear combination of falling factorials using the Stirling number interpretation for the coefficients. Your final answer should have actual numbers, and should not use the notation $S(n,k)$.

(2) Let n be a positive integer. Find simple formulas for $c(n+2,n)$ and $c(n+3,n)$.

(3) Below, "evaluate" means give a simple formula that only involves n.
 (a) Evaluate $\displaystyle\sum_{i=0}^{n}\binom{n}{i}4^{n-i}$.
 (b) Evaluate $\displaystyle\sum_{i=0}^{n}\binom{n}{i}(-1)^i 4^{n-i}$.
 (c) Evaluate $\displaystyle\sum_{\substack{0\le i\le n\\ i \text{ even}}}\binom{n}{i}4^{n-i}$ (the sum is over i from 0 to n such that i is even).
 (d) Evaluate $\displaystyle\sum_{\substack{0\le i\le n\\ i \text{ even}}} i\binom{n}{i}4^{n-i}$ (the sum is over i from 0 to n such that i is even).

(4) Fix non-negative integers n, m, k. Explain why the coefficient of x^k in $(x+1)^n\cdot(x+1)^m$ is

$$\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}.$$

Explain why this fact implies the following formula:

$$\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}=\binom{n+m}{k}.$$

(5) If you use formula from class, state which one you are using.
 (a) What is the coefficient of $x_1^4x_3^2x_4^3$ in $(x_1+x_2+x_3-x_4-x_5)^9$?
 (b) What is the coefficient of x^{13} in $\dfrac{1}{(2-x)^2}$?
 (c) Let α be a scalar. What is the coefficient of x^{11} in $\dfrac{1+2x^3}{(1+\alpha x)^4}$?
 (d) Let α be a scalar. What is the coefficient of x^3 in $\dfrac{\sqrt{1-\alpha x}}{(1-x)^5}$?

Checking your work

Here I'm going to give you some points of data that you can use to check if your final answer is correct. Usually, the form of your answer will look like a complicated expression and it's fine to leave it like that. These are provided to help make sure you can solve the problems *correctly* without spoiling the solution, and I encourage the use of computers to check your work.

(2) You should have a formula involving n. For $n = 6$, it should evaluate to $c(8, 6) = 322$ and $c(9, 6) = 4536$.

(3) You should have a formula involving n. For $n = 6$, it should evaluate to: (a) 15625 (b) 729 (c) 8177 (d) 8646

(5) For last parts, answer is a formula involving α. (a) -1260 (b) approximately .000427 (c) For $\alpha = 1$, should get -34 (d) For $\alpha = 1$, should get 26.8125

Hints

You will generally get more out of the homework if you try to think about how to solve the problems by yourself without any help. However, it's much better to solve the problem with help than to not solve it at all. So if you get stuck, try reading below. Beyond that, you can ask other students or ask the professor/TAs in discussion, office hours, or Discord.

(2) Similar to HW2 #4, but see also Example 3.3.3.

(3) See Section 4 of notes, especially the results derived from binomial theorem and Example 4.1.4.

(4) Use the definition of multiplication of polynomials (or if you like, treat them as power series and use the definition we gave).

(5) These involve rewriting in a way that's more convenient to use and applications of general binomial theorem. Try using these:

(b): $(2 - x) = 2(1 - \frac{x}{2})$

(c): $(1 + \alpha x)^{-4} + 2x^3(1 + \alpha x)^{-4}$

(d): $(1 - \alpha x)^{1/2}(1 - x)^{-5}$