Explanations should be given for your solutions. Use complete sentences. Some hints are on the last page.

(1) Evaluate the following sums:
   (a) \( \sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^i} \)
   (b) \( \sum_{i=0}^{n} i \binom{n}{i} 3^i \)

(2) Fix positive integers \( n, m, k \). Prove that
   \[
   \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}.
   \]

(3) Let \( n \geq 2 \) be an integer.
   (a) Prove that
   \[
   \sum_{i=0}^{n} i \binom{n}{i} (-1)^{i-1} = 0.
   \]
   (b) Deduce from (a) that
   \[
   \sum_{0 \leq i \leq n} i \binom{n}{i} = \sum_{0 \leq i \leq n} i \binom{n}{i}
   \]
   and compute the common value.

(4) (a) Using the multinomial theorem, compare the coefficients of both sides of the equation \( (x + y + z)(x + y + z)^n = (x + y + z)^{n+1} \) to get a generalization of Pascal’s identity for multinomial coefficients.
   (b) Do the same thing with \( k \) variables for general \( k \).

(5) A “forward path” in the plane is a sequence of steps of the form \((1, 0)\) and \((0, 1)\).
   (a) How many forward paths are there from \((0, 0)\) to \((a, b)\) where \(a, b\) are non-negative integers?
   (b) Let \( S_{a,b} \) be the set of integer partitions \( \lambda \) such that \( \ell(\lambda) \leq b \) and \( \lambda_1 \leq a \). Find a bijection between \( S_{a,b} \) and the set of forward paths from \((0, 0)\) to \((a, b)\).
   (c) Generalize this definition to \( d \) dimensions by only allowing steps which increase one of the coordinates by 1 (so \((1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1))\). How many forward paths are there from \((0, 0, \ldots, 0)\) to \((a_1, a_2, \ldots, a_d)\) where \(a_1, \ldots, a_d\) are non-negative integers?
Hints:
5b: Draw a rectangle with endpoints (0,0), (a,0), (a,b), (0,b). Think of a forward path as splitting this rectangle into two pieces and consider the portion above the path.