Math 184, Fall 2023
Homework 4
Due: Wednesday, Nov. 8 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences. **I'll put some hints and answers to check your work on the next page.**

(1) You can use a computer algebra system to solve a system of linear equations. If you do this, say so; you should still explain how you set up the problem.
  (a) Find a closed formula for a_n, which is defined by
$$a_0 = 1, \qquad a_1 = 1, \qquad a_n = 2a_{n-1} + 3a_{n-2} \qquad \text{for } n \geq 2.$$
  (b) Find a closed formula for b_n, which is defined by
$$b_0 = 1, \qquad b_1 = 3, \qquad b_n = 8b_{n-1} - 16b_{n-2} + 2 \cdot 3^n \qquad \text{for } n \geq 2.$$
  (c) Find a closed formula for c_n, which is defined by
$$c_0 = 1, \qquad c_n = 5c_{n-1} + 4n \qquad \text{for } n \geq 1.$$

(2) Define a sequence by
$$a_0 = -1, \qquad a_1 = 3, \qquad a_2 = 1, \qquad a_n = 3a_{n-2} + 2a_{n-3} + n^2 \qquad \text{for } n \geq 3.$$
Write $A(x) = \sum_{n \geq 0} a_n x^n$ as a rational function (= a polynomial divided by another polynomial) in x. You do not need to solve for a closed formula for a_n.

(3) Let $S(n, k)$ be the Stirling number of the second kind. For each $k \geq 1$, define the ordinary generating function
$$\mathbf{S}_k(x) = \sum_{n \geq 0} S(n,k) x^n = S(0,k) + S(1,k)x + S(2,k)x^2 + \cdots$$
  (a) For $k \geq 2$, translate the identity from lecture
$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$$
  into an identity involving $\mathbf{S}_k(x)$ and $\mathbf{S}_{k-1}(x)$.
  (b) Use the identity you found in (a) and induction on k to show that for all $k \geq 1$:
$$\mathbf{S}_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

(4) Let u_n be the number of integer partitions of n that only use the parts 3, 4, and 5 such that 5 can only be used at most once (i.e., $m_5(\lambda) \leq 1$ and $m_k(\lambda) = 0$ if $k \notin \{3, 4, 5\}$).
  (a) Write $U(x) = \sum_{n \geq 0} u_n x^n$ as a rational function.
  (b) Use (a) to get a homogeneous linear recurrence relation for u_n. Make sure to state the relevant initial conditions.

(5) For $n > 0$, let a_n be the number of integer partitions of n such that every part appears at most twice, and let b_n be the number of integer partitions of n such that no part is divisible by 3. Show that $a_n = b_n$ for all n.

Checking your work

Here I'm going to give you some points of data that you can use to check if your final answer is correct. Usually, the form of your answer will look like a complicated expression and it's fine to leave it like that. These are provided to help make sure you can solve the problems *correctly* without spoiling the solution, and I encourage the use of computers to check your work.

(1) You should have a formula involving n. If you plug in $n = 6$, you should get (a) 365 (b) 47938 (c) 35149

(2) Your answer should be in the form $p(x)/q(x)$ where p, q are polynomials. For my answer, I got $p(2) = 3$ and $q(2) = 27$. Depending on how you wrote it, you may get something different, but the ratio should be 1/9 at $x = 2$.

(4) If you plug in values into your recurrence for $n = 12$, you should get $u_{12} = 3$.

Hints

You will generally get more out of the homework if you try to think about how to solve the problems by yourself without any help. However, it's much better to solve the problem with help than to not solve it at all. So if you get stuck, try reading below. Beyond that, you can ask other students or ask the professor/TAs in discussion, office hours, or Discord.

(2) To deal with the n^2 term, use the following observation: given a formal power series $B(x) = \sum_{n\geq 0} b_n x^n$, we have $xB'(x) = \sum_{n\geq 0} n b_n x^n$.

(3) Similar to the type of computation done in the proof of Theorem 6.1.4: start with definition of $\mathbf{S}_k(x)$ and plug in the identity and simplify the result.

(4) Examples 6.2.9 and 6.2.12 are helpful here, and a useful identity is $1 + y + y^2 = \frac{1-y^3}{1-y}$.

(5) Generalize the proof of Theorem 6.2.11.