Explanations should be given for your solutions. Use complete sentences.

(1) If \( \sum_{n \geq 0} a_n x^n = \frac{1 - x^2 + 2x^3}{(1-3x)^4} \), find a formula for the \( a_n \).

(2) Define a sequence by
\[ a_0 = 1, \quad a_1 = 3, \quad a_n = 8a_{n-1} - 16a_{n-2} \quad \text{for } n \geq 2. \]

(a) Express \( A(x) = \sum_{n \geq 0} a_n x^n \) as a rational function in \( x \).

(b) Find a closed formula for \( a_n \).

(3) You want to build a stack of blocks that is \( n \) feet high. You have 5 different kinds (unlimited of each): red and blue blocks are 1 foot high, while green, yellow, and orange blocks are 2 feet high. Let \( a_n \) be the number of ways to stack these blocks.

(a) Find a linear recurrence relation and initial conditions satisfied by \( a_n \).

(b) Find a closed formula for \( a_n \).

(4) You are designing a race that takes place over \( n \) blocks in a city. It will consist of 3 portions: running, followed by biking, and ending with another running portion. The end of a portion should match up with the end of a block. The first running portion needs to designate 2 blocks to have a first aid tent, and the biking portion needs to designate 3 blocks to have a first aid tent. The second running portion doesn’t need anything, but must have positive length. Use generating functions to find a formula for the number of ways to design a race under these conditions.

(5) Let \( n \) be a positive integer and let \( a_n \) be the number of different ways to pay \( n \) dollars using only 1, 2, 5, 10, 20 dollar bills in which at most four 10 dollar bills are used. Define \( a_0 = 1 \). Express \( A(x) = \sum_{n \geq 0} a_n x^n \) as a rational function.

(6) Let \( S(n, k) \) be the Stirling number of the second kind. For each \( k \geq 1 \), define the ordinary generating function
\[ S_k(x) = \sum_{n \geq 0} S(n, k)x^n. \]

(a) For \( k \geq 2 \), translate the identity from class
\[ S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k) \]
into an identity involving \( S_k(x) \) and \( S_{k-1}(x) \).

(b) Use the identity you found in (a) and induction on \( k \) to show that for all \( k \geq 1 \):
\[ S_k(x) = \frac{x^k}{(1-x)(1-2x) \cdots (1-kx)}. \]