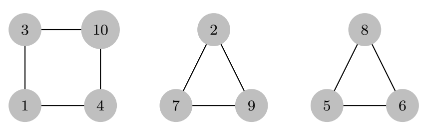
Math 184, Fall 2023
Homework 6
Due: Wednesday, Dec. 6 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences.

(1) For $n > 0$, let h_n be the number of permutations of size n that only have cycles of even length and define $h_0 = 1$.
  (a) Find a structure α so that $h_n = |e^\alpha([n])|$.
  (b) Apply the derivative identity (Proposition 7.2.2) to get a recurrence relation for h_n. Be sure to give the relevant initial conditions.

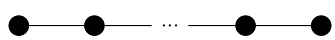
(2) In this problem, we will consider labeled graphs such that every vertex has degree 2 (i.e., is contained in exactly 2 edges). For $n \geq 1$, let h_n be the number of such graphs with vertex set $[n]$ and set $h_0 = 1$.

You may use this fact without proving it: every graph satisfying this condition is a disjoint union of cycle graphs, each one consisting of 3 or more vertices. If our vertex set is $[10]$, one example is



  (a) Find a structure α so that $h_n = |e^\alpha([n])|$.
  (b) Apply the derivative identity (Proposition 7.2.2) to get a recurrence relation for h_n. Be sure to give the relevant initial conditions.

(3) Define a path graph to be a graph consisting of 2 or more vertices that looks like this:



For $n \geq 1$, let h_n be the number of labeled graphs with vertex set $[n]$ which are a disjoint union of labeled path graphs and set $h_0 = 1$.
  (a) Find a structure α so that $h_n = |e^\alpha([n])|$.
  (b) Apply the derivative identity (Proposition 7.2.2) to get a recurrence relation for h_n. Be sure to give the relevant initial conditions.

(4) In Example 7.3.2, we counted the number of labeled trees on 3 and 4 vertices by considering all types of unlabeled trees and counting how many different labelings each one has. Do the same for 5 and 6 vertices.

(5) Use Lagrange inversion to solve these problems:
  (a) Let $A(x) = \sum_{n\geq 0} a_n x^n$ be the formal power series satisfying the identity
$$A(x) = \frac{x}{(2 - A(x))^4}.$$
Find a closed formula for a_n.
  (b) Let $B(x) = \sum_{n\geq 0} b_n x^n$ be the formal power series satisfying the identity
$$B(x) = 2 + 3xB(x)^5.$$
Find a closed formula for b_n.

Checking your work

For problems 1, 2, 3, I will list the derivative of $E_\alpha(x)$:

Q1: $(E_\alpha(x))' = \dfrac{x}{1-x^2}$, Q2: $(E_\alpha(x))' = \dfrac{x^2}{2(1-x)}$, Q3: $(E_\alpha(x))' = \dfrac{x(2-x)}{2(1-x)^2}$.

Q4: Your answers should add up to $5^3 = 125$ and $6^4 = 1296$, respectively.

Q5: Your answer should be a formula in n. For $n = 4$, this should specialize to: (a) $\frac{51}{131072}$ (b) 3025797120

Hints

Q1: Examples 7.2.3, 7.2.5, 7.2.6 use the same idea.

Q2: A cycle graph is very similar to a cycle in a permutation, *except* there is no orientation, i.e., there is no difference between clockwise and counterclockwise.

For applying the derivative identity, pay attention to how it is used in Example 7.2.6 to "clear denominators".

Q3: This is similar to Q2, so do that one first.

Q4: There are 3 types of unlabeled trees on 5 vertices and 6 types of unlabeled trees on 6 vertices.

Q5: For part b, this is not in the right form, see Example 7.4.2 for how to modify it.