Math 184, Final exam
Instructor: Steven Sam
June 13, 2023
8AM – 11AM

Your name:
(Try to use the exact name that is in Gradescope, since it will be automatically matched.)

Student ID:

- **No books, materials, notes, cell phones, calculators, etc.** Consulting other students or any other sources is considered an **academic integrity violation** and will be treated as such.
  A list of some selected formulas is provided on the back of this sheet.
- Pages will be separated for scanning. **Write your name at the top of each page.** Also, make sure to write **legibly** and **dark enough** and **not too close to the edges of the paper**.
- By default, write your answers only in the space provided. The extra blank sheets can be used for your solution, but **clearly indicate** in the problem if you want the extra sheets to be graded.
- Cross out / erase irrelevant scratch work. If you write incorrect statements without crossing them out, you may lose points. **Make clear what your final answer is.**
- Answers should always have explanations. **You may lose points otherwise.**
- If you finish early, double-check your work and make sure you followed the above instructions. When you're ready, you may turn it in and leave.
- To turn in exam, show your ID and make sure your name is checked off the list.

Good luck!

Formula sheet

- The number of k-element multisets of an n-element set is $\binom{n+k-1}{k}$.
- The number of weak compositions of n with k parts is $\binom{n+k-1}{k-1}$.
- The change of basis between powers and falling factorials is

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k$$

  where $S(n,k)$ is the Stirling number.

- If d, n are non-negative integers, then

$$\binom{-d}{n} = (-1)^n \binom{d+n-1}{n}.$$

- Lagrange inversion formula: if $G(x)$ is a formal power series with nonzero constant term, then there is a unique formal power series $A(x)$ such that

$$A(x) = xG(A(x)).$$

  Furthermore, $A(x)$ has no constant term and for $n \geq 1$, we have

$$[x^n]A(x) = \frac{1}{n}[x^{n-1}](G(x)^n).$$

- Given an alphabet of size k, the number of words of period d is

$$\omega(d) = \sum_{e|d} \mu(d/e)k^e,$$

  where μ is the Möbius function, and the number of necklaces of length n is

$$\sum_{d|n} \frac{\omega(d)}{d}.$$

Name:

1. (10 pts) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 184$$

   if the x_i are all required to be positive even integers?

2. (10 pts) Let $B(x) = \sum_{n\geq 0} b_n x^n$ be the formal power series satisfying the identity

$$B(x) = 1 + 2xB(x)^{-3}.$$

   Find a closed formula for b_n.

3. (10 pts) We have n distinguishable tables. We want to paint each one either red, blue, or green such that #(red tables) + #(blue tables) is even. How many ways can this be done?

4. (10 pts) For $n > 0$, let h_n be the number of set partitions of $[n]$ such that there are no blocks of size 7; set $h_0 = 1$. Give a simple expression for the EGF $H(x) = \sum_{n \geq 0} \frac{h_n}{n!} x^n$.

Name:

5. (10 pts) Given a standard deck of 52 cards, how many ways are there to choose 9 cards so that we have 3 pairs and a triple (a pair means 2 cards with the same value and a triple means 3 cards with the same value; we also require that we have 3 different pairs, i.e., no "4 of a kind").

6. (10 pts) Evaluate $\displaystyle\sum_{i=0}^{n} i^2 \binom{n}{i} 2^i 3^{n-i}$.

7. (15 pts) How many ways can we pick subsets S_1, S_2, S_3, S_4 of $[n]$ so that

$$S_1 \subsetneqq S_2 \subsetneqq S_3 \subsetneqq S_4?$$

Here "$X \subsetneqq Y$" means that X is a subset of Y and that $X \neq Y$.

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8. (10 pts) Using an alphabet of size k, how many words of period 100 are there?

9. (15 pts) Let $n \geq 3$ be an integer. Find a simple formula for the Stirling number $S(n+3, n)$. *Do NOT use the inclusion-exclusion formula!*

10. (15 pts) Let u_n be the number of integer partitions of n that only use the parts 2 and 5.

(a) Write $U(x) = \sum_{n\geq 0} u_n x^n$ as a rational function (= ratio of two polynomials in x).

(b) Use your answer in (a) to get a linear recurrence relation for u_n. Make sure to state the relevant initial conditions.

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11. (20 pts) Given a subset $S \subseteq [n]$, call it "spaced out" if any two different elements are at least 3 apart from each other (in symbols: if $i, j \in S$ and $i \neq j$, then $|i - j| \geq 3$). For example, all of the spaced out subsets of [5] are

$$\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 4\}, \{1, 5\}, \{2, 5\}.$$

Let g_n be the number of spaced out subsets of $[n]$.

(a) For $n \geq 3$, prove that $g_n = g_{n-1} + g_{n-3}$. What are the initial conditions?

Part (b) is on the next page.

(b) From (a): $g_n = g_{n-1} + g_{n-3}$ for $n \geq 3$.
Express $G(x) = \sum_{n\geq 0} g_n x^n$ as a rational function (= ratio of two polynomials in x).

Name:

Extra scratch paper. If you want this space graded, clearly say so in the problem that you are working on so we know to look here.

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