Twisted homological stability for groups via functor categories

Steven V Sam
(joint work with Andrew Putman)

A sequence of groups and maps $G_1 \to G_2 \to \cdots$ satisfies **homological stability** if, for each $i \geq 0$, the induced map on homology $H_i(G_n) \to H_i(G_{n+1})$ is an isomorphism for $n \gg i$. Some sequences of groups that satisfy homological stability (the maps are the usual ones):

- Symmetric groups $G_n = S_n$ (Nakaoka [Nak]);
- For any group $\Gamma$, the wreath products $G_n = S_n \rtimes \Gamma^n$ (this seems to have been well-known – it is stated explicitly in [HW, Prop. 1.6]);
- For well-behaved rings $R$ (such as commutative noetherian rings of finite Krull dimension), $G_n = GL_n(R)$ (van der Kallen [Va]), and
- the symplectic groups $G_n = Sp_{2n}(R)$ (Mirzaii–van der Kallen [MV]).

More generally, $G_n$-representations $M_n$ equipped with $G_n$-equivariant maps $M_n \to M_{n+1}$ satisfy **twisted homological stability** if, for each $i \geq 0$, the induced map $H_i(G_n; M_n) \to H_i(G_{n+1}; M_{n+1})$ is an isomorphism for $n \gg i$.

The problem we consider is to determine which kinds of sequences satisfy twisted homological stability. Wahl [W] gave a general setup using the notion of **homogeneous categories** (they are monoidal categories; we omit the definition since we use a special case below). If $(G, \oplus, 0)$ is a symmetric monoidal groupoid such that $\text{Aut}(0) = \{1\}$ and such that the map $\text{Aut}(A) \to \text{Aut}(A \oplus B)$ given by $f \mapsto f \oplus 1_B$ is injective for all $A, B$, then there is a minimal homogeneous symmetric monoidal category $UG$ containing $G$ as its underlying groupoid [W, 1.4, 1.5].

Corresponding to the previous examples, we give a few cases of $G$ and $UG$:

- The groupoid of finite sets under disjoint union gives the category FI, whose objects are finite sets and whose morphisms are injections;
- The groupoid of free $\Gamma$-sets under disjoint union gives the category $\text{FI}\Gamma$, whose objects are finite sets and whose morphisms are $\Gamma$-injections: an injective function $f : R \to S$ and a function $\rho : R \to \Gamma$; the composition with $(g : S \to T, \sigma)$ is given by $(gf, \tau)$ where $\tau(x) = \sigma(f(x)) \cdot \rho(x)$;
- The groupoid of finite rank free $R$-modules under direct sum gives the category $\text{VIC}(R)$, whose objects are finite rank free $R$-modules and whose morphisms $V \to W$ are pairs of maps $V \to W \to V$ composing to $1_V$;
- The groupoid of finite rank free symplectic $R$-modules under direct sum gives the category $\text{SI}(R)$, whose objects are finite rank free symplectic $R$-modules and whose morphisms are linear maps preserving the form (and hence must be injective).

The above examples of $UG$ are in fact complemented categories. A symmetric monoidal category is **complemented** if it satisfies the following properties:

- Every morphism is a monomorphism;
- $0$ is an initial object, and so we have canonical maps $V \to V \oplus V'$ and $V' \to V \oplus V'$;
The map \( \text{Hom}(V \oplus V', W) \to \text{Hom}(V, W) \times \text{Hom}(V', W) \) is injective;

- Every subobject \( C \subseteq V \) has a complement, i.e., another subobject \( D \subseteq V \) so that \( V \cong C \oplus D \) and where the isomorphism identifies the inclusion \( C \subseteq V \) with the canonical map \( C \to C \oplus D \), and similarly for \( D \).

Each one has a **generator** \( X \), i.e., every object is isomorphic to \( X^{\oplus n} \).

- Fix a commutative ring \( k \). Given a complemented category \( C \) with generator \( X \), and a functor \( F : C \to k\text{-Mod} \), define \( \Sigma F : C \to k\text{-Mod} \) to be the precomposition with the functor \( Y \mapsto Y \oplus X \). There is a natural transformation \( F \to \Sigma F \), and its kernel and cokernel are denoted \( \ker F \) and \( \text{coker} F \). We can use this to define the **degree** of a functor:
  - If \( F = 0 \), then its degree is \(-1\);
  - If \( \ker F \) and \( \text{coker} F \) have degree \( \leq r - 1 \), then \( F \) has degree \( \leq r \).

Otherwise \( F \) has infinite degree. Also, for each \( n \), define a semisimplicial set \( W_n(X) \) whose \( p \)-simplices are \( \text{Hom}(X^{\oplus p+1}, X^{\oplus n}) \).

Let \( C \) be a complemented category with generator \( X \). Suppose that there is an integer \( k \geq 2 \) so that for all \( n \geq 1 \), \( W_n(X) \) is \((n - 2)/k\)-connected. Then a special case of [W, Theorem 5.6] is that for any functor of finite degree \( r \), the map

\[ H_i(\text{Aut}(X^{\oplus n}); F(X^{\oplus n})) \to H_i(\text{Aut}(X^{\oplus n+1}); F(X^{\oplus n+1})) \]

is an isomorphism when \( i \leq (n - r)/k \). Implicitly, we always use the morphisms \( X^{\oplus n} \to X^{\oplus n+1} \) as inclusion via the first \( n \) factors to define all structure maps. We will say that the functor \( F \) satisfies homological stability.

For some purposes, having finite degree is too restrictive of a condition. For example, if \( k \) is a field and \( F \) takes finite-dimensional values, then it implies that the function \( n \mapsto \dim_k F(X^{\oplus n}) \) is a polynomial for \( n \gg 0 \). A basic property of complemented categories \( C \) with generator \( X \) is that for \( n \geq r \), the permutation representation \( k[\text{Hom}(X^{\oplus r}, X^{\oplus n})] \) is isomorphic to the induced representation \( \text{Ind}_{\text{Aut}(X^{\oplus r})}^{\text{Aut}(X^{\oplus n})} k \). So by Shapiro’s lemma, the functor \( P_r : C \to k\text{-Mod} \) defined by \( Y \mapsto k[\text{Hom}(X^{\oplus r}, Y)] \) satisfies homological stability if the same is true for the constant functor, i.e., the groups \( \text{Aut}(X^{\oplus n}) \) satisfy homological stability. **From now on, we will make this assumption about \( \text{Aut}(X^{\oplus n}) \).**

By Yoneda’s lemma, the set of natural transformations \( P_r \to F \) identifies with \( F(X^{\oplus r}) \), and so the \( P_r \) are a set of projective generators for the functor category \([C, k\text{-Mod}]\). In particular, any functor \( F \) admits a projective resolution of the form

\[ \cdots \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to F \to 0 \]

where \( P_d \) is a direct sum of \( P_r \). If we assume that each \( P_d \) has a decomposition as \( \bigoplus_{r \leq d} P_r \) (\( D \) depending on \( d \)), then \( P_d \) also satisfies homological stability. Note that for each \( n \), there is a spectral sequence

\[ E^1_{p,q}(n) = H_p(\text{Aut}(X^{\oplus n}); P_q(X^{\oplus n})) \Rightarrow H_{p+q}(\text{Aut}(X^{\oplus n}); F(X^{\oplus n})), \]

and spectral sequence morphisms \( E^1_{p,q}(n) \to E^1_{p,q}(n + 1) \). So with the assumption on \( P_d \) above, we see that for a given diagonal \( p + q \), the map of spectral sequences

\[ E_d^1(n) \to 0 \]

for all \( d \). Therefore, the spectral sequence converges to something and we have a decomposition of \( F(X^{\oplus r}) \).
on all relevant terms to calculate $H_{p+q}$ is an isomorphism for $n \gg 0$, and hence $F$ satisfies homological stability.

This motivates the following definitions. Say that $F$ is **finitely generated** if it is a quotient of a finite direct sum $P_{r_1} \oplus \cdots \oplus P_{r_n}$, and say that $F$ is **noetherian** if every subfunctor of $F$ is finitely generated; $[C, k\text{-Mod}]$ is (locally) noetherian if every finitely generated functor is noetherian. This implies that $k$ is a noetherian ring. If $[C, k\text{-Mod}]$ is noetherian, then every finitely generated functor has a projective resolution where each $P_d$ is a finite direct sum of $P_r$, and hence satisfies homological stability. This is formalized in [PS, Theorem 4.2].

Some examples of when $[C, k\text{-Mod}]$ is noetherian (take $k$ to be any noetherian ring) corresponding to the running examples:

- FI (Church–Ellenberg–Farb–Nagpal [CEFN, Theorem A])
- When $\Gamma$ is virtually polycyclic, FI$_\Gamma$ (Sam–Snowden [SS, Cor. 1.2.2])
- When $R$ is a finite commutative ring, VIC($R$) and SI($R$) (Putman–Sam [PS, Theorems C, D])

Finally, a word about cohomology versus homology. Let $k$ be a field of characteristic $p > 0$ and let $h(n) = \{(x_1, \ldots, x_n) \in k^n \mid \sum_i x_i = 0\}$ be the reflection representation of $S_n$; note that $\{1, \ldots, n\} \mapsto h(n)$ defines a finitely generated functor $FI \to k\text{-Mod}$. For $n \geq 3$ we have $H_0(S_n; h(n)) = 0$, whereas

$$H^0(S_n; h(n)) = h^{S_n} = \begin{cases} 0 & \text{if } p \nmid n \\ k & \text{if } p \mid n \end{cases}.$$  

In fact, this periodic behavior is typical: Nagpal shows that if $F$ is a finitely generated FI-module, then for each $i$, the function $n \mapsto \dim_k H^i(S_n; F(\{1, \ldots, n\}))$ is a periodic function of $n$ for $n \gg 0$ with period a power of $p$ [Nag, Theorem D].

**References**


