1. Measures

1.1. Introduction.

1.2. $\sigma$-algebras.

1.3. Measures

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3.1. Signed Measures

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3.5. Functions of Bounded Variation

4. General Techniques

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TODO Add problems from midterm/final? If do so be sure to ask for Adrian's permission to use them if decide to post on the internet.

1. Measures

1.1. Introduction.

1.2. $\sigma$-algebras.

1.1. What does it mean for $\mathcal{A}$ to be an algebra of sets on $X$? How about a $\sigma$-algebra? Give a few examples. (**Hint:**

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1.2. If $E$ is a set, what is $\mathcal{M}(E)$? (Hint: )

1.3. If $X$ is a metric space, what is its corresponding Borel $\sigma$-algebra $\mathcal{B}_X$? (Hint: )

1.4. Prove that if $F \subseteq M$ where $\mathcal{M}$ is a $\sigma$-algebra, then $\mathcal{M}(F) \subseteq M$, and that if $E \subseteq F$, then $\mathcal{M}(E) \subseteq \mathcal{M}(F)$.

1.5. Prove that $\mathcal{B}_R = \mathcal{M}(E_1)$ where $E$ consists of all open intervals of $\mathbb{R}$. (Hint: )

1.5.1. Prove the same result if one replaces $E$ with the set of closed intervals (or intervals of the form $[a,b)$, or intervals of the form $(a,b])$. (Hint: )

1.6. For $\{X_\alpha\}_{\alpha \in A}$, each with the associated $\sigma$-algebra $\{\mathcal{M}_\alpha\}$, we define the product $\sigma$-algebra $\bigotimes \mathcal{M}_\alpha$ on $\prod X_\alpha$ to be generated by $\{\pi_\alpha^{-1}(E) : \alpha \in A, E \in \mathcal{M}_\alpha\}$, where $\pi_\alpha^{-1} = \Pi E_\beta$ with $E_\beta = E_\alpha$ if $\beta = \alpha$ and $E_\beta = X_\beta$ otherwise (i.e. the preimage of $E_\alpha$ under the projection map onto $X_\alpha$). This is analogous to the direct sum of groups $\bigoplus A_\alpha$ defined by having only finite positions non-zero, i.e. this is the coproduct of the $\sigma$-algebras in the category theory sense of the word.

Prove that if $A$ is countable, then $\bigotimes \mathcal{M}_\alpha$ is generated by $\{\Pi E_\alpha : \alpha \in A, E_\alpha \in \mathcal{M}_\alpha\}$. Prove that this is not necessarily the case when $A$ is uncountable. (Hint: )

1.7. Assume $E_\alpha$ generates $\mathcal{M}_\alpha$. Prove that $F_1 = \{\pi_\alpha^{-1}(E_\alpha) : \alpha \in A, E_\alpha \in E_\alpha\}$ generates $\bigotimes \mathcal{M}_\alpha$. Further, if $A$ is countable and $X_\alpha \in E_\alpha$ for all $\alpha$, prove that $F_2 = \{\Pi E_\alpha : E_\alpha \in E_\alpha\}$ generates $\bigotimes \mathcal{M}_\alpha$. (Hint: )

1.8. Let $\mathcal{B}_{X_i}$ be the borel sets for some metric spaces $X_i$ and let $X = \prod^n X_i$. Prove that $\bigotimes^n_i \mathcal{B}_{X_i} \subseteq \mathcal{B}_X$. Moreover, if each $X_i$ is separable (i.e. there exists a dense countable set $C_i$ in each $X_i$), prove that $\bigotimes^n_i \mathcal{B}_{X_i} = \mathcal{B}_X$. Conclude that $\mathcal{B}_R^n = \bigotimes^n_i \mathcal{B}_R$. (Hint: )

1.9. Here’s a technical definition/lemma that will be useful later. Define an elementary family to be a collection $\mathcal{E}$ of subsets of $X$ such that (1) $\emptyset \in \mathcal{E}$, (2) $E, F \in \mathcal{E} \implies E \cap F \in \mathcal{E}$, (3) if $E \in \mathcal{E}$ then $E^c$ is a finite disjoint union of members of $\mathcal{E}$. 
1.3. Measures.

1.10. What does it mean for \((X, \mathcal{M}, \mu)\) to be a measured space? Specifically, what properties does \(\mu\) satisfy? (\textbf{Hint}: )

1.11. \textbf{[Monotonicity]} Prove that if \(F \subseteq E\), then \(\mu(F) \leq \mu(E)\).

1.12. \textbf{[Subadditivity]} Prove that \(\mu(\bigcup E_i) \leq \sum \mu(E_i)\). (\textbf{Hint}: )

1.13. \textbf{[Continuity from below]} Prove that if \(E_1 \subseteq E_2 \cdots\) then \(\mu(\bigcup E_i) = \mu(\lim E_i) = \lim \mu(E_i)\). (\textbf{Hint}: )

1.14. \textbf{[Continuity from above]} Prove that if \(E_1 \supset E_2 \cdots\) and \(\mu(E_1) < \infty\) then \(\mu(\bigcap E_i) = \lim \mu(E_i)\). (\textbf{Hint}: )

1.15. A measure \(\mu\) is said to be finite on \(X\) if \(\mu(X) < \infty\), it is said to be \(\sigma\)-finite if there exists countably many (not necessarily disjoint) \(\{E_i\}\) such that \(X = \bigcup E_i\) with \(\mu(E_i) < \infty\) for all \(i\), and it is said to be semifinite if for all \(E\) such that \(\mu(E) = \infty\) there exists an \(F \subseteq E\) such that \(0 < \mu(F) < \infty\).

Prove that any \(\sigma\)-finite measure is semifinite, but that there exists semifinite measures that are not \(\sigma\)-finite. (\textbf{Hint}: )

1.16. If \(\mu\) is a semifinite measure and \(\mu(E) = \infty\), prove that for any \(C > 0\) there exists \(F \subseteq E\) with \(C < \mu(F) < \infty\). (\textbf{Hint}: )

1.17. A set \(E \in \mathcal{M}\) is said to be a null set if \(\mu(E) = 0\). If \(F \subseteq E\) and \(F \in \mathcal{M}\), then \(\mu(F) = 0\), but it need not be the case that \(F \in \mathcal{M}\). A measure such that all subsets of null sets are measurable is said to be complete. Not all measures are complete, but all measures can be (uniquely) completed.

Let \(\mathcal{N}\) be the set of subsets of null sets. Define \(\mathcal{M} = \{E \cup F | E \in \mathcal{M}, F \in \mathcal{N}\}\) and \(\bar{\mu}(E \cup F) = \mu(E)\). Prove that \(\mathcal{M}\) is a \(\sigma\)-algebra and that
\( \bar{\mu} \) is a (well-defined) measure on \( \bar{\mathcal{M}} \), and that it is the only measure on \( \bar{\mathcal{M}} \) that extends \( \mu \). (Hint: ) (Hint: ) (Hint: )

1.18. Let \( (X, \mathcal{M}, \mu) \) be a measure space and \( \{E_n\} \subseteq \mathcal{M} \) such that \( \mu(E_m \Delta E_n) \to 0 \) as \( m, n \) to infinity. Prove that there exists an \( E \) such that \( \mu(E \Delta E_n) \to 0 \). (Hint: ) (Hint: ) (Hint: )

1.19. An outer measure is a function \( \mu^* : \mathcal{P}(X) \to [0, \infty] \) satisfying what three properties? (Hint: )

1.20. Construct an outer measure where \( \mu^*(\bigcup A_i) = \sum \mu^*(A_i) \) does not always hold when the \( A_i \) are disjoint. (Hint: )

1.21. If \( \mu^* \) is an outer measure on \( X \) then \( A \subseteq X \) is said to be \( \mu^* \)-measurable if 
\[ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \] for all \( E \subseteq X \). Note that this is equivalent to saying \( \mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \) for all \( E \subseteq X \), since the other direction is automatic from subadditivity.

Motivation: say we have a set of measurable sets \( \mathcal{M} \) with \( \mu^* \) as the measure. If we want to add \( A \) to \( \mathcal{M} \), then it better satisfy \( \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \) for all \( E \in \mathcal{M} \). Thus, if \( A \) is \( \mu^* \)-measurable we can always add it to our base set \( \mathcal{M} \) without having to worry what exactly \( \mathcal{M} \) is.

[Carathéodory’s Theorem] If \( \mu^* \) is an outer measure on \( X \), then the collection \( \mathcal{M} \) of \( \mu^* \)-measurable sets is a \( \sigma \)-algebra, and the restriction of \( \mu^* \) to \( \mathcal{M} \) is a complete measure.

1.21.1. First, prove that \( \mathcal{M} \) is an algebra by showing that it is closed under compliments, and that \( A \cup B \) is \( \mu^* \)-measurable whenever \( A, B \) are. (Hint: )

1.21.2. Prove that \( \mu^* \) is finitely additive on \( \mathcal{M} \), i.e. if \( A, B \in \mathcal{M} \) are disjoint, then \( \mu^*(A \cup B) = \mu^*(A) + \mu^*(B) \).

1.21.3. Prove that \( \mathcal{M} \) is closed under countable unions, making it a \( \sigma \)-algebra. (Hint: ) (Hint: )
1.21.4. Prove that $\mu^*$ is countably additive on $\mathcal{M}$, making it a measure. (Hint: )

1.21.5. Prove that $\mu^*$ is complete. (Hint: )

1.22. Given an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$, we’d like to extend it to a $\sigma$-algebra. To this end, we say that $\mu_0 : \mathcal{A} \to [0, \infty]$ is a premeasure if $\mu_0(\emptyset) = 0$ and whenever $\{A_i\}$ are disjoint sets in $\mathcal{A}$ such that $\bigcup_{i=1}^\infty A_i \in \mathcal{A}$, then $\mu_0(\bigcup A_i) = \sum \mu_0(A_i)$.

1.22.1. Let $\mathcal{E} \subseteq \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset, X \in \mathcal{E}$ and $\rho(\emptyset) = 0$ (not necessarily a premeasure). For any $A \subseteq X$, define

$$\rho^*(A) = \inf \{\sum \rho(E_j) : E_j \in \mathcal{E}, A \subseteq \bigcup E_j\}.$$

Prove that $\rho^*$ is an outer measure. (Hint: )

1.22.2. By 1.22.1., every premeasure induces an outer measure defined by

$$\mu^*(E) = \inf \{\sum_{i=1}^\infty \mu_0(A_j) : A_j \in \mathcal{A}, E \subseteq \bigcup A_j\}.$$

Prove that (1) $\mu^*$ restricted to $\mathcal{A}$ is $\mu_0$, and (2) every set in $\mathcal{A}$ is $\mu^*$ measurable. (Hint: ) (Hint: )

1.22.3. Let $\rho$ and $\rho^*$ be as in 1.22.1. with $\mathcal{E} = \mathcal{A}$ an algebra. Prove that $\rho$ is a premeasure iff (1) $\rho^*$ restricted to $\mathcal{A}$ is $\rho$, and (2) every set in $\mathcal{A}$ is $\rho^*$-measurable. (Hint: ) (Hint: )

This explains why the definition for premeasure is what it is: it’s the weakest restriction for a function $\rho : \mathcal{A} \to [0, \infty]$ to induce an “appropriate” measure and $\sigma$-algebra from $\mathcal{A}$.

1.22.4. TODO What conditions need to be imposed on a $\rho : \mathcal{E} \to [0, \infty]$ to guarantee that $\rho^*(E) = \rho(E)$ and all the $E \in \mathcal{E}$ are $\rho^*$-measurable? Well from the proof of the above statement a necessary condition is $\rho(E) = \sum \rho(E_i)$ whenever $\sqcup E_i = E$, but to get sufficiency it looks like we really need to use the properties of $\mathcal{A}$ being a premeasure.

1.23. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, $\mu_0$ a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the $\sigma$-algebra generated by $\mathcal{A}$.

1.23.1. Prove that there exists a measure $\mu$ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is $\mu_0$, namely $\mu = \mu^*|_{\mathcal{M}}$ where $\mu^*$ is defined as above. (Hint: 1.21. 1.22.2.)
1.23.2. If \( \nu \) is another measure on \( \mathcal{M} \) that extends \( \mu_0 \), prove that \( \nu(E) \leq \mu(E) \) for all \( E \in \mathcal{M} \). (Hint: )

1.23.3. Prove that \( \nu(A) = \mu(A) \) for \( A = \bigcup A_i, \ A_i \in \mathcal{A} \). (Hint: )

1.23.4. Prove that \( \nu(E) = \mu(E) \) whenever \( \mu(E) < \infty \). TODO

1.23.5. Prove that if \( \mu_0 \) is \( \sigma \)-finite, then \( \mu \) is the unique extension of \( \mu_0 \) to a measure \( \mathcal{M} \). (Hint: )

For the case \( \mathcal{A} \) = the algebra of intervals and \( \mu_0 \) defined to be the usual length on intervals, this statement says that there’s a unique measure on \( \mathcal{B}_\mathbb{R} \) such that the intervals have their natural measure, which we will call the Lebesgue measure.

1.24. Let \( \mathcal{A} \subseteq \mathcal{P}(X) \) be an algebra, \( \mathcal{A}_\sigma \), the collection of countable unions of sets in \( \mathcal{A} \) and \( \mathcal{A}_{\sigma \delta} \) the collection of countable intersection of sets in \( \mathcal{A}_\sigma \). Let \( \mu_0 \) be a premeasure on \( \mathcal{A} \) and \( \mu^* \) the induced outer measure.

1.24.1. For any \( E \subseteq X \) and \( \epsilon > 0 \), prove that there exists \( A \in \mathcal{A}_\sigma \) with \( E \subseteq A \) and \( \mu^*(A) \leq \mu^*(E) + \epsilon \).

1.24.2. If \( \mu^*(E) < \infty \), prove that \( E \) is \( \mu^* \)-measurable iff there exists \( B \in \mathcal{A}_{\sigma \delta} \) with \( E \subseteq B \) and \( \mu^*(B \setminus E) = 0 \). (Hint: )

1.24.3. If \( \mu_0 \) is \( \sigma \)-finite, prove that the restriction \( \mu^*(E) < \infty \) in the previous part is superfluous.

1.25. This exercise shows that the measure produced at the end of 1.22.2. is as “full” as possible, i.e. we can’t repeat the procedure and get a larger \( \sigma \)-algebra.

Let \( \mu^* \) be an outer measure on \( X \), \( \mathcal{M}^* \) the \( \sigma \)-algebra of \( \mu^* \)-measurable sets \( \bar{\mu} = \mu^*|\mathcal{M}^* \), and \( \mu^+ \) the outer measure induced by \( \bar{\mu} \) as in 1.22.2. (with \( \bar{\mu} \) and \( \mathcal{M}^* \) replacing \( \mu_0 \) and \( \mathcal{A} \)).

1.25.1. If \( E \subseteq X \), prove that we have \( \mu^*(E) \leq \mu^+(E) \), with equality iff there exists \( A \in \mathcal{M}^* \) with \( A \supseteq E \) and \( \mu^*(A) = \mu^*(E) \). (Hint: 1.23.2. )

1.25.2. If \( \mu^* \) is induced from a premeasure, prove that \( \mu^* = \mu^+ \).

1.25.3. If \( X = \{0, 1\} \), prove that there exists an outer measure \( \mu^* \) on \( X \) such that \( \mu^* \neq \mu^+ \). (Hint: )

1.5. Borel Measures on the Real Line.
1.26. Let $\mu$ be a finite Borel measure on $\mathbb{R}$ (i.e. a measure on the $\sigma$-algebra $\mathcal{B}_\mathbb{R}$). Define the distribution function of $\mu$ to be $F(x) = \mu(((-\infty, x])$. Prove that $F$ is increasing, right continuous, and that $\mu((a, b]) = F(b) - F(a)$.

Note that we chose $\mu$ to be finite to avoid complications. In the more general setting we’ll be given any increasing, right continuous $F$ and construct a measure $\mu$ from $F$ via $\mu((a, b]) := F(b) - F(a)$.

1.27. Define an h-interval (h for half-open) to be an interval of the form $(a, b]$ or $(a, \infty)$. Accept that these sets are an elementary family, and hence an algebra by 1.9.

Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. If $(a_j, b_j]$ are disjoint h-intervals let

$$
\mu_0\left(\bigcup_{i=1}^n (a_j, b_j]\right) = \sum [F(b_j) - F(a_j)],
$$

and let $\mu_0(\emptyset) = 0$.

1.27.1. Prove that $\mu_0$ is well defined (i.e. if you write $(a, b]$ as a finite union of disjoint h-intervals, whichever representation you apply $\mu_0$ to you get the same answer). (Hint: First prove the result when the union is just an interval.) (Hint: For the general case of $\bigcup_{i} I_i = \bigcup_{j} J_j$ refine both sets to $\bigcup_{i} I_i \cap J_j$.)

1.27.2. Prove that $\mu_0$ is a premeasure on $\mathcal{A}$. SKIP

1.28. Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing right continuous function.

1.28.1. Prove that there is a unique Borel measure $\mu_F$ on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all $a, b$. (Hint: 1.27. 1.23.5.)

1.28.2. Let $G$ be another such function. Prove that $\mu_F = \mu_G$ iff $F - G$ is constant.

1.28.3. Conversely, if $\mu$ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets, we can define $F(x) = \mu((0, x])$ if $x > 0$, $F(x) = -\mu((x, 0]$) if $x < 0$, and $F(0) = 0$. Prove/observe that $F$ is increasing, right continuous, and $\mu = \mu_F$.

1.29. Given a distribution $F$ and its associated measure $\mu_F$, we can look at its completion $\bar{\mu}_F$, which we’ll denote simply by $\mu$, with domain $\mathcal{M}_\mu$. By definition this means for all $E \in \mathcal{M}_\mu$ we have

$$
\mu(E) = \inf\left(\sum [F(b_j) - F(a_j)] : E \subseteq \bigcup(a_j, b_j]\right) = \inf\left(\sum \mu((a_j, b_j]) : E \subseteq \bigcup(a_j, b_j]\right).
$$

Prove that we can replace h-intervals by open h-intervals in that definition, i.e.

$$
\mu(E) = \inf\left\{\sum \mu((a_j, b_j]) : E \subseteq \bigcup(a_j, b_j]\right\}.
$$

Note that you can’t just say “$\mu((a, b]) = \mu((a, b))$”, since there could be a point mass at $b$ (i.e. $F$ is the indicator function that we’re $\geq b$). SKIP
1.30. If $E \in \mathcal{M}_\mu$, prove that
\[
\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open}\} = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\}.
\]

(Hint: \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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2.2. If \((X, \mathcal{M})\) is a measurable space and \(f\) is a real valued function, then we’ll say the function is \(\mathcal{M}\)-measurable if it’s \((\mathcal{M}, \mathcal{B}_\mathbb{R})\)-measurable. Prove that \(f\) being \(\mathcal{M}\)-measurable is equivalent to saying \(f^{-1}((a, \infty)) \in \mathcal{M}\) for all \(a \in \mathbb{R}\) (or any other family of infinite half intervals).

2.3. We’ll say that a function \(f : \mathbb{R} \to \mathbb{C}\) is Lebesgue/Borel measurable if it is \((\mathcal{L}, \mathcal{B}_\mathbb{C})/ (\mathcal{B}_\mathbb{R}, \mathcal{B}_\mathbb{C})\)-measurable, and similarly define this for \(f : \mathbb{R} \to \mathbb{R}\) (thus Borel measurable is a stronger condition).

Prove that if \(f : \mathbb{R} \to \mathbb{R}\) is monotonic, then \(f\) is Borel measurable.

2.4. Given a set \(X\), a family of measurable spaces \(\{(Y_\alpha, \mathcal{N}_\alpha)\}\), and functions \(f_\alpha : X \to Y_\alpha\), there is a unique “smallest” \(\sigma\)-algebra on \(X\) such that each \(f_\alpha\) is measurable, namely the \(\sigma\)-algebra generated by \(f_\alpha^{-1}(E)\) for all \(E \in \mathcal{N}_\alpha\). Note that if \(X = \prod Y_\alpha\), \(f_\alpha = \pi_\alpha\), then the \(\sigma\)-algebra we get is precisely the product \(\sigma\)-algebra as defined before.

Let \(Y = \prod Y_\alpha\), \(\mathcal{N} = \bigotimes \mathcal{N}_\alpha\), \(\pi_\alpha\) the projection maps, and \((X, \mathcal{M})\) a measurable space. Prove that \(f : X \to Y\) is \((\mathcal{M}, \mathcal{N})\)-measurable iff \(f_\alpha = \pi_\alpha f\) is \((\mathcal{M}, \mathcal{N}_\alpha)\)-measurable for all \(\alpha\).

2.5. Prove that a function \(f : X \to \mathbb{C}\) is \(\mathcal{M}\)-measurable iff \(\Re f\) and \(\Im f\) are \(\mathcal{M}\)-measurable. \(\text{Hint:}\)

2.6. If \(f, g : X \to \mathbb{C}\) are \(\mathcal{M}\)-measurable, prove that \(f + g\) and \(fg\) are also \(\mathcal{M}\)-measurable. \(\text{Hint:}\)

2.7. Define Borel sets on the extended reals \(\mathbb{R}\) by \(\mathcal{B}_\mathbb{R} = \{E : E \cap \mathbb{R} \in \mathcal{B}_\mathbb{R}\}\), in particular this means intervals of the form \((a, \infty]\) generate the Borel sets. We’ll say a function \(f : X \to \mathbb{R}\) is \(\mathcal{M}\)-measurable if it is \((\mathcal{M}, \mathcal{B}_\mathbb{R})\)-measurable.

Let \(\{f_j\}\) be a sequence of \(\mathbb{R}\)-valued measurable functions on \((X, \mathcal{M})\). Prove that the following functions are all measurable:

2.7.1. \(g_1(x) = \sup f_j(x)\). Prove that this result need not hold if the supremum is taken over uncountably many functions. \(\text{Hint:}\)

2.7.2. \(g_2(x) = \inf f_j(x)\).

2.7.3. \(g_3(x) = \lim \sup f_j(x)\). \(\text{Hint:}\)

2.7.4. \(g_4(x) = \lim \inf f_j(x)\).

2.7.5. Further, if \(f(x) = \lim f_j(x)\) exists for all \(x \in X\), then \(f\) is measurable. \(\text{Hint:}\)

2.8. Use the previous exercise to prove that \(\max(f, g)\) and \(\min(f, g)\) are measurable whenever \(f, g\) are. In particular, conclude that \(f^+ = \max(f, 0)\) and \(f^- = \max(f, 0)\) are measurable whenever \(f\) is.

2.9. We now discuss characteristic functions and simple functions.
2.9.1. The characteristic function $\chi_E$ is defined to be 1 on $E$ and 0 elsewhere. Prove/observe that $\chi_E$ is measurable iff $E$ is.

2.9.2. A simple function is a finite linear combination of characteristic functions of sets in $\mathcal{M}$. Prove equivalently that $f : X \to \mathbb{C}$ is simple iff $f$ is measurable and the range of $f$ is a finite subset of $\mathbb{C}$. In particular, 

$$f = \sum z_j \chi_{E_j}, \ E_j = f^{-1}(\{z_j\}), \ range(f) = \{z_1, \ldots, z_n\},$$

which is called the standard representation of $f$ (exhibiting $f$ as a linear combination with distinct coefficient of characteristic functions of disjoint sets whose union is $X$).

2.10. If $f : X \to [0, \infty]$ is measurable, prove that there is a sequence of simple functions $\{\phi_n\}$ such that $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f, \phi_n \to f$ pointwise and the convergence is uniform on any set on which $f$ is bounded. (Hint: )

2.10.1. Further prove that if $f : X \to \mathbb{C}$ is measurable then there is a sequence of simple functions $\{\phi_n\}$ such that $0 \leq |\phi_1| \leq \cdots \leq |f|$ and the guys converge pointwise to $f$ and uniformly on any bounded interval.

2.11. If a statement about points is true for all $x \notin N, \ \mu(N) = 0$, then we say the statement is true almost everywhere, and sometimes we say it’s true $\mu$-a.e.

Prove that the following statements are valid iff $\mu$ is complete.

2.11.1. If $f$ is measurable and $f = g$ $\mu$-a.e., then $g$ is measurable. (Hint: )

2.11.2. If $f_n$ is measurable and $f_n \to f$ $\mu$-a.e., then $f$ is measurable.

2.12. Let $(X, \mathcal{M}, \mu)$ be a measure space and $(X, \mathcal{M}, \bar{\mu})$ its completion. If $f$ is an $\mathcal{M}$-measurable function on $X$, prove that there is an $\mathcal{M}$-measurable function $g$ such that $f = g \bar{\mu}$-a.e. (i.e. if you’re careless about whether $\mu$ is complete or not it’s not a big deal). TODO

2.13. Let $(X, \mu)$ be a measure space and $f, f_n : X \to \mathbb{R}$ be measurable functions such that $f_1 \leq \cdots \leq f_n \leq \cdots$ a.e. and $\lim f_n = f$ a.e.

2.13.1. Prove that for every $a \in \mathbb{R}$, $\lim \mu(\{x : f_n(x) > a\})$ exists and is equal to $\mu(\{x : f(x) > a\})$.

2.13.2. Assume that $\mu(X) < \infty$. Show that for every $a \in \mathbb{R}$, $\lim \mu(\{x : f_n(x) < a\})$ exists and that it satisfies

$$\mu(\{x : f(x) < a\}) \leq \lim \mu(\{x : f_n(x) < a\}) \leq \mu(\{x : f(x) < a\}) + \mu(\{x : f(x) = a\}).$$

Give an example where each inequality is strict. (Hint: )

2.2. Integration of Nonnegative Functions.
2.14. For a fixed measure space \((X, \mathcal{M}, \mu)\), we define \(L^+\) to be the space of measurable functions from \(X\) to \([0, \infty]\).

If \(\phi\) is a simple function in \(L^+\) with standard representation \(\phi = \sum a_j \chi_{E_j}\), we define the integral of \(\phi\) with respect to \(\mu\) to be

\[
\int \phi d\mu = \sum a_j \mu(E_j),
\]

where we say \(0 \cdot \infty = 0\) if it comes up in that sum.

Note that \(\phi \chi_A\) is simple if \(A \in \mathcal{M}\), so we define \(\int_A \phi d\mu = \int \phi \chi_A d\mu\). When it is notational convenient we will sometimes use \(\int \phi\) or \(\int \phi(x) d\mu(x)\) instead of \(\int \phi d\mu\).

Prove the following facts for \(\phi, \psi\) simple functions in \(L^+\).

2.14.1. \(\int c\phi = c \int \phi\) if \(c \geq 0\).

2.14.2. \(\int (\phi + \psi) = \int \phi + \int \psi\).

2.14.3. If \(\phi \leq \psi\) then \(\int \phi \leq \int \psi\).

2.14.4. The map \(A \to \int_A d\mu\) is a measure on \(\mathcal{M}\) (namely, prove that if \(A = \bigcup A_k\) disjoint, then \(\int_A \phi = \sum\int_{A_k} \phi\)). \((\text{Hint:})\)

2.15. We define the integral for any function \(f \in L^+\) by taking

\[
\int f d\mu = \sup\{\int \phi d\mu : 0 \leq \phi \leq f, \phi \text{ simple}\}.
\]

2.15.1. Prove that this definition agrees with the previous definition when \(f\) is simple.

2.15.2. Prove/observe that \(\int f \leq \int g\) if \(f \leq g\) and \(\int cf = c \int f\) for all \(c \geq 0\).

2.16. If \(f \in L^+\) and \(\int f < \infty\), prove that \(\{x : f(x) = \infty\}\) is a null set and that \(\{x : f(x) > 0\}\) is \(\sigma\)-finite. \((\text{Hint:})\)

2.17. [Monotone Convergence Theorem] If \(\{f_n\}\) is a sequence in \(L^+\) such that \(f_j \leq f_{j+1}\) for all \(j\) and \(f = \lim f_n (= \sup f_n)\), then \(\int f = \lim \int f_n\). \((\text{Hint:})\)

One useful aspect of this is that it reduces the computation of \(\int f\), which originally involved taking the supremum of a huge set, to computing \(\lim \int \phi_n\) whenever \(\phi_n\) converge to \(f\) from below, which we know exists by 2.10. \((\text{Hint:})\)

2.17.1. Prove that the hypothesis of the theorem that \(\{f_n\}\) be increasing (at least a.e.) is essential by considering \(\chi_{[n,n+1]}\) or \(n\chi_{(0,1/n)}\) with \(\mu = m\).
2.18. Prove using MCT that if \( \int f_1 < \infty \), \( f_j \geq f_{j+1} \), and \( f_n \to f \), then \( \int f = \lim \int f_n \). \((\text{Hint:})\)

There’s another, less direct but somewhat interesting, way of proving this statement, which also gives some insight in how to prove later problems. Observe that we can assume \( f = 0 \) and that \( f_1 \) (and hence all \( f_n \)) is finitely valued.

2.18.1. Prove the statement when \( \mu(X) < \infty \) and \( f_1 \) is bounded by some value \( M \). \((\text{Hint:})\)

2.18.2. Prove the statement in general. \((\text{Hint:})\)

2.19. If \( \{f_n\} \) is a finite or infinite sequence in \( L^+ \) and \( f = \sum f_n \), prove that \( \int f = \sum \int f_n \).

2.20. If \( f \in L^+ \) then \( \int f = 0 \) iff \( f = 0 \) a.e. \((\text{Hint:})\) \((\text{Hint:})\)

2.21. If \( \{f_n\} \subseteq L^+ \), \( f \in L^+ \) and \( f_n(x) \) increases to \( f(x) \) for a.e. \( x \), prove that \( \int f = \lim \int f_n \). \((\text{Hint:})\) \((\text{Hint:})\) \(2.20.\)

2.22. [Fatou’s Lemma] If \( \{f_n\} \subseteq L^+ \) prove that

\[
\int (\lim \inf f_n) \leq \lim \inf \int f_n.
\]

\((\text{Hint:})\) \((\text{Hint:})\) \((\text{Hint:})\)

The intuition for this is that as your \( n \) increases you can have mass move around, but as your limit goes to infinity you can only lose mass by having it all collapse to a small set or having it move out to infinity. Verify that this is what happens for \( \chi_{[n,n+1]} \) or \( n \chi_{(0,1/n)} \) with \( \mu = m \).

2.23. If \( \{f_n\} \subseteq L^+ \), \( f \in L^+ \) and \( f_n \to f \) a.e., prove that \( \int f \leq \lim \inf \int f_n \). \((\text{Hint:})\) \(2.22. \ 2.21.\)

2.24. Suppose \( \{f_n\} \subseteq L^+ \), \( f_n \to f \) pointwise, and \( \int f = \lim \int f_n < \infty \). Prove that \( \int_E f = \lim \int_E f_n \) for all \( E \in \mathcal{M} \), but that this need not hold if \( \int f = \lim \int f_n = \infty \). \text{TODDO}

2.3. Integration of Complex Functions.

2.25. We define the integral of \( \int f \) for \( f \) a real-valued function to be \( \int f^+ - \int f^- \), provided both of these values are finite.

Prove that the set of integrable real-valued functions on \( X \) is a real vector space and that the integral is a linear functional on it.
2.26. If $f$ is a complex-valued measurable function then we will say that it is
integrable if $\int |f| < \infty$. Since $|f| \leq |\Re f| + |\Im f| \leq 2|f|$, $f$ is integrable iff
$\Re f, \Im f$ are both integrable, and in this case we define $\int f = \int \Re f + i \int \Im f$.
One can similarly observe that the space of complex integrable functions is
a complex vector space and the integral is a linear functional. Denote this
space by $L^1(\mu), L^1(X, \mu), L^1(X)$, or simply $L^1$ depending on context.

Prove that if $f \in L^1$, then $|\int f| \leq \int |f|$ (the statement is nearly trivial
when $f$ is real; the difficulty is when $f$ is complex valued).

2.27. Prove that if $f \in L^1$, then $\{x : f(x) \neq 0\}$ is $\sigma$-finite. (Hint: 2.16.)

2.28. If $f, g \in L^1$, prove that $\int_E f = \int_E g$ for all $E \in \mathcal{M}$ iff
$\int |f - g| = 0$ iff $f = g$ a.e. (Hint: 2.20.) (Hint: )

Because of the above result, we shall consider the elements of $L^1$ as the
equivalence classes of a.e.-defined integrable functions on $X$ where $f \sim g$
iff $f = g$ a.e. The central benefit to this is that we can make $L^1$ a metric
space with metric $\rho(f, g) = \|f - g\|_1 = \int |f - g|$ (the central point being we
need $\rho(f, g) = 0$ to hold only when $f, g$ are in the same equivalence class).
We shall refer to convergence with respect to this metric as convergence in
$L^1$, i.e. $f_n \to f$ in $L^1$ iff $\int |f_n - f| \to 0$.

2.29. [Dominated Convergence Theorem] Let $\{f_n\} \subseteq L^1$ be such that $f_n \to f$
a.e. and such that there exists a non-negative $g \in L^1$ such that $|f_n| \leq g$
a.e. for all $n$. Prove that $f \in L^1$ and $\int f = \lim \int f_n$. (Hint: 2.20.

The intuition behind the theorem is that if we can confine the graphs of
the $|f_n|$ to a region of the plane with finite area, then the area beneath it
can’t escape to infinity and we get convergence.

2.30. Prove directly (i.e. without appealing to DCT or its proof) that if $\mu(X) <
\infty$ and $f_n \in L^1, f_n \to f, |f_n| \leq 1$ for all $n$, then $\int f_n \to \int f$. (Hint:

2.31. Suppose that $\{f_j\}$ is a sequence in $L^1$ such that $\sum |f_j| < \infty$. Then
$\sum f_j$ converges a.e. to a function in $L^1$ and $\int \sum f_j = \sum \int f_j$. (Hint:

2.32. Let $f \in L^1, \epsilon > 0$.

2.32.1. Prove that there is an integrable simple function $\phi = \sum a_j \chi_{E_j}$ such
that $\int |f - \phi|d\mu < \epsilon$, i.e. the integrable simple functions are dense
in \( L^1 \) under its metric. \( \text{(Hint: } \)\)

2.32.2. Further, if \( \mu \) is a Lebesgue-Stieltjes measure on \( \mathbb{R} \) then the sets \( E_j \) can be taken to be finite unions of open intervals. \( \text{(Hint: } \)\)

\( 2.36. \) Give examples of the following. \( \text{(Hint: } \)\)

\( 2.37. \) We’ll say that \( f \) is Cauchy in measure if for all \( \epsilon > 0 \),

\[ \mu(\{ x : h|f_n(x) - f_m(x)| \geq \epsilon \}) \to 0. \]

for \( n, m \) sufficiently large.

Prove that \( f_n \to f \) in measure implies \( f_n \) is Cauchy in measure. It’s not as obvious that the converse holds as well.

\( 2.38. \) If \( f_n \) is Cauchy in measure, prove that there is a measurable function \( f \) such that \( f_n \to f \) in measure, and that there is a subsequence \( f_{n_j} \) that converges
to \( f \) a.e. Moreover, if \( f_n \to g \) in measure, then \( g = f \) a.e. (Hint: \( \mu \))

(2.16)

2.38. Conclude that if \( f_n \to f \) in \( L^1 \), then there is a subsequence \( f_{n_j} \) such that \( f_{n_j} \to f \) a.e.

2.39. [Egoroff’s Theorem] Suppose that \( \mu(X) < \infty \) and \( f_n \) are a sequence of measurable functions such that \( f_n \to f \) a.e. Prove that for every \( \epsilon > 0 \) there exists \( E \subseteq X \) such that \( \mu(E) < \epsilon \) and \( f_n \to f \) uniformly on \( E^c \). (Hint:)

(2.39.1) Conclude that if \( f_n \to f \) a.e. and \( \mu(X) < \infty \), then \( f_n \to f \) in measure.

(2.39.2) Prove that Egoroff’s Theorem still holds if one replaces “\( \mu(X) < \infty \)” with “\( |f_n| \leq g \) for all \( n \) for some \( g \in L^1(\mu) \).” (Hint:)

The moral of the proof of this exercise is that when \( \mu(X) < \infty \), often your argument involves splitting \( X \) into a bad part (whose measure goes to 0 faster than the “measure of badness”) and a good part (whose measure of badness goes to 0 at any rate, which is fine because the good part has measure at most \( \mu(X) < \infty \)).

From this point on for this subsection there’s just a bunch of random exercises (possibly too many). Feel free to move along whenever you want to.

2.40. Suppose \( \mu(X) < \infty \). If \( f, g \) are complex-valued measurable functions on \( X \), define \( \rho(f, g) = \frac{|f - g|}{1 + |f - g|} \). Assume that \( \rho \) is a metric on the space of measurable functions (provided we identify functions that are equal a.e.).

Prove that \( f_n \to f \) with respect to this metric iff \( f_n \to f \) in measure. (Hint:)

2.41. Prove that if \( f_n \geq 0 \) and \( f_n \to f \) in measure, then \( \int f \leq \lim \inf \int f_n \). (Hint:)

2.38. (Hint:)

2.42. Suppose \( |f_n| \leq g \in L^1 \) and \( f_n \to f \) in measure.
2.42.1. Prove that \( \int f = \lim \int f_n \). (Hint: 2.41.)

2.42.2. Prove that \( f_n \to f \) in \( L^1 \). (Hint: )

2.43. Find a sequence \( f_n \) such that \( f_n \to f \) pointwise, \( \int f_n \to \int f \), but such that \( f_n \) does not converge to \( f \) in \( L^1 \). (Hint: )

2.44. Prove that if \( \mu(E_n) < \infty \) for \( n \in \mathbb{N} \) and \( \chi_{E_n} \to f \) in \( L^1 \), then \( f \) is (a.e. equal to) the characteristic function of a measurable set. (Hint: )

2.45. Suppose \( f_n \to f \) in measure and \( g_n \to g \) in measure.

2.45.1. Prove that \( f_n + g_n \to f + g \) in measure.

2.45.2. Prove that \( f_n g_n \to fg \) in measure if \( \mu(X) < \infty \), but not necessarily if \( \mu(X) = \infty \). (Hint: )

2.46. [Lusin’s Theorem] Prove that if \( f : [a, b] \to \mathbb{C} \) is Lebesgue measurable and \( \epsilon > 0 \), then there is a compact set \( E \subseteq [a, b] \) such that \( \mu(E^c) < \epsilon \) and \( f|E \) is continuous. (Hint: 2.32. 2.38.1.)

The rest of the problems here are sample midterm questions. They don’t fit in this subsection perfectly, but they’re very good problems and they have a similar “feel” to the problems in this subsection.

2.47. Let \( f_n \geq 0 \) and assume \( \int f_n \to 0 \). Prove that we need not have \( f_n \to 0 \), but that there always exists a subsequence such that \( f_{n_k} \to 0 \) \( \mu \)-a.e. (Hint: 2.36.2.) (Hint: ) (Hint: )

2.48. Assume \( \mu(X) < \infty \). Let \( f_n : X \to \mathbb{R} \) be measurable functions. Prove that there exists a sequence of real numbers \( c_n \) such that \( \frac{f_n(x)}{c_n} \to 0 \) for \( \mu \)-a.e. \( x \in X \). (Hint: ) (Hint: )
2.49. Let \((X, \mathcal{M}, \mu)\) be a measure space. Let \(\{f_n\} \subseteq L^1(\mu)\) s.t. \(\int |f_{n+1} - f_n|d\mu < 2^{-n}\) for all \(n \geq 1\). Prove that \(f(x) = \lim f_n(x)\) exists for a.e. \(x \in X\), \(f \in L^1(\mu)\), and \(\|f_n - f\|_1 \to 0\). (Hint:)

2.50. We want to talk about the product of spaces \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\).

2.50.1. Conclude that if \(E \subseteq \mathcal{M} \otimes \mathcal{N}\), then \(\mu(E) = \sum \mu(A_j)\nu(B_j)\). (Hint:)

2.50.2. Suppose that \(A \times B\) is a rectangle that is the finite/countable disjoint union of rectangles \(A_j \times B_j\). Prove/observe that \(\chi_A(x)\chi_B(y) = \sum \chi_{A_j}(x)\chi_{B_j}(y)\).

2.50.3. In the context of the previous problem, prove that \(\mu(A)\nu(B) = \sum \mu(A_j)\nu(B_j)\). (Hint:)

2.50.4. Conclude that if \(E\) is the disjoint union of finitely many rectangles \(A_j \times B_j\) then \(\pi(E) = \sum \mu(A_j)\nu(B_j)\) is well defined. (Hint:)

Note that \(\pi\) is a premeasure on \(\mathcal{A}\), so it generates an outer measure whose restriction to \(\mathcal{M} \otimes \mathcal{N}\) is a measure denoted \(\mu \times \nu\). Moreover, if \(\mu, \nu\) are \(\sigma\)-finite, then it’s not hard to see that \(\mu \times \nu\) is also \(\sigma\)-finite, and hence by 1.23, it’s the unique measure on \(\mathcal{M} \otimes \mathcal{N}\) such that \(\mu \times \nu(A \times B) = \mu(A)\nu(B)\) for all rectangles \(A \times B\).

2.51. This problem deals with sections.

2.51.1. Let \(E_x = \{y \in Y : (x, y) \in E\}\), \(E^y = \{x \in X : (x, y) \in E\}\) be the \(x\)-section and \(y\)-section of \(E\) respectively. Prove that if \(E \subseteq \mathcal{M} \otimes \mathcal{N}\), then \(E_x \subseteq \mathcal{N}\) and \(E^y \subseteq \mathcal{M}\) for all \(x, y\). (Hint:)

2.51.2. If \(f\) is a function on \(X \times Y\), define \(f_x(y) = f^y(x) = f(x, y)\), the \(x\)-section and \(y\)-section of \(f\) respectively. Prove that if \(f\) is \(\mathcal{M} \otimes \mathcal{N}\)-measurable, then \(f_x\) is \(\mathcal{N}\)-measurable for all \(x \in X\) and \(f^y\) is \(\mathcal{M}\)-measurable for all \(y \in Y\). (Hint:)

)
2.52. We define a monotone class of a space $X$ to be a subset $C \subseteq \mathcal{P}(X)$ that is closed under countable increasing unions and countable decreasing intersections (if $\{E_j\} \subseteq C$ and $E_j \subseteq E_{j+1}$, then $\bigcup E_j \in C$, similar for intersections).

Clearly every $\sigma$-algebra is a monotone class, and the intersection of a family of monotone classes is a monotone class. Because of the latter, for any $E \subseteq \mathcal{P}(X)$ there is a unique smallest monotone class containing $E$ called the monotone class generated by $E$.

[The Monotone Class Lemma] If $\mathcal{A}$ is an algebra of subsets of $X$, then the monotone class $C$ generated by $\mathcal{A}$ coincides with the $\sigma$-algebra $\mathcal{M}$ generated by $\mathcal{A}$.

2.52.1. Define $C(E) = \{F \in C : E \setminus F, F \setminus E, E \cap F \in C\}$. Prove/observe that $\emptyset, E \in C(E)$, $F \in C(E) \iff E \in C(F)$, and that $C(E)$ is a monotone class.

2.52.2. If $E \in \mathcal{A}$ prove that $F \in C(\mathcal{A})$ for all $F \in \mathcal{A}$. Conclude that $\mathcal{A} \subseteq C(E)$ and hence $C \subseteq C(E)$ in this case.

2.52.3. Further conclude that $C \subseteq C(F)$ for all $F \in C$. (Hint: $\mu \times \nu(E_x) = \int \nu(E_x d\mu(x) = \int \mu(E^y) d\nu(y)$.

2.52.4. Conclude that $C$ is an algebra. (Hint: $\mu \times \nu(E_x) = \int \nu(E_x d\mu(x) = \int \mu(E^y) d\nu(y)$.

2.52.5. Conclude that $C$ is a $\sigma$-algebra. (Hint: $\mu \times \nu(E_x) = \int \nu(E_x d\mu(x) = \int \mu(E^y) d\nu(y)$.

2.53. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are $\sigma$-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$ then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable on $X$ and $Y$ respectively and

$$\mu \times \nu(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y).$$

2.53.1. Suppose that $\mu, \nu$ are finite. Let $C$ denote the set of all $E \in \mathcal{M} \otimes \mathcal{N}$ such that the conclusions for the theorem is true. Prove that all rectangles $A \times B$ are in $C$. (Hint: $\mu \times \nu(E_x) = \int \nu(E_x d\mu(x) = \int \mu(E^y) d\nu(y)$.

2.53.2. Conclude that finite disjoint unions of rectangles are in $C$, and hence it’s sufficient to prove that $C$ is a monotone class by 2.52..

2.53.3. Prove that if $\{E_n\}$ is an increasing sequence in $C$, then $E = \bigcup E_n \in C$. (Hint: $\mu \times \nu(E_x) = \int \nu(E_x d\mu(x) = \int \mu(E^y) d\nu(y)$.

2.53.4. Prove that if $\{E_n\}$ is a decreasing sequence in $C$, then $E = \bigcap E_n \in C$. (Hint: $\mu \times \nu(E_x) = \int \nu(E_x d\mu(x) = \int \mu(E^y) d\nu(y)$.

2.53.5. TODO The above work shows that the result is true for finite measure spaces. Conclude the result for $\sigma$-finite measure spaces.
2.54. [Fubini-Tonelli] Suppose that \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) are \(\sigma\)-finite measure spaces. Recall that \(L^+(X)\) is the space of all measurable functions from \(X\) to \([0, \infty]\).

2.54.1. [Tonelli] Prove if \(f \in L^+(X \times Y)\), then the functions \(g(x) = \int f_x \, d\nu\) and \(h(y) = \int f^y \, d\mu\) are in \(L^+(X)\) and \(L^+(Y)\) respectively, and further that
\[
\int f \, (\mu \times \nu) = \int \left[ \int f(x,y) \, d\nu(y) \right] \, d\mu(x) = \int \left[ \int f(x,y) \, d\mu(x) \right] \, d\nu(y).
\]
(Hint: 2.53.)(Hint: )

2.54.2. [Fubini] If \(f \in L^1(\mu \times \nu)\), prove that \(f_x \in L^1(\nu)\) for a.e. \(x \in X\), \(f^y \in L^1(\mu)\) for a.e. \(y \in Y\), the a.e. defined functions \(g(x) = \int f_x \, d\nu\), \(h(y) = \int f^y \, d\mu\) are in \(L^1(\mu), L^1(\nu)\), and the integration equalities for the previous result holds. (Hint: )

2.54.3. Interpret this last result for the case \(X = Y = \mathbb{N}\) and \(\mu = \nu = \) the counting measure.

2.55. All of the hypothesis in the Fubini-Tonelli theorem are necessary. In the exercises below, compute each of the integrals, observe that the values are all different, and note which hypothesis of the Fubini-Tonelli theorem doesn’t hold.

2.55.1. Let \(X = Y = [0,1]\), \(\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}\), \(\mu\) the Lebesgue measure and \(\nu\) the counting measure. Let \(D = \{(x,x) : x \in [0,1]\}\), the diagonal in \(X \times Y\). Prove that \(\int \int \chi_D \, d\mu \, d\nu\), \(\int \int \chi_D \, d\nu \, d\mu\), and \(\int \chi_D \, (\mu \times \nu)\) are all unequal. (Hint: )

2.55.2. Let \(X = Y\) be an uncountable linearly ordered set such that for each \(x \in X\), \(\{y \in X : y < x\}\) is countable. Let \(\mathcal{M} = \mathcal{N}\) be the \(\sigma\)-algebra of countable or co-countable sets and let \(\mu = \nu\) be defined on \(\mathcal{M}\) by \(\mu(A) = 0\) if \(A\) is countable and \(\mu(A) = 1\) if \(A\) is co-countable. Let \(E = \{(x,y) : y < x\}\). Prove that \(E_x, E^y\) are measurable for all \(x, y\) and compute \(\int \int \chi_E \, d\mu \, d\nu\), \(\int \int \chi_E \, d\nu \, d\mu\). (Hint: )

2.55.3. Let \(X = Y = \mathbb{N}\), \(\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})\), \(\mu = \nu\) the counting measure. Define \(f(m,n) = 1\) if \(m = n\), \(f(m,n) = -1\) if \(m = n + 1\) and \(f(m,n) = 0\) otherwise. Prove that \(\int |f| \, (\mu \times \nu) = \infty\) and compute
\[ \int \int f \, d\mu \, d\nu, \quad \int \int f \, d\nu \, d\mu. \quad (\text{Hint:}) \]

2.56. We deal with the issue of completeness.

2.56.1. Let \( \mathcal{M}, \mathcal{N} \) be complete measures with \( \emptyset \neq A \in \mathcal{M}, \mu(A) = 0, \) and \( \mathcal{N} \neq \mathcal{P}(Y) \). Prove that \( \mu \times \nu \) is not complete, i.e. even if you have two complete measures it’s very unlikely that their product is complete. (\text{Hint:})

2.56.2. An obvious solution to the above problem is to just take the completion of the given measure, but when we do so the relationship between the measurability of a function on \( X \times Y \) and its measurability on its \( x, y \)-sections is not so simple. However, we do still get Fubini-Tonelli after making some suitable adjustments.

Let \( (X, \mathcal{M}, \mu) \) and \( (Y, \mathcal{N}, \nu) \) be complete \( \sigma \)-finite measure spaces and \( (X \times Y, \mathcal{L}, \lambda) \) be the completion of the measure \( \mu \times \nu \). If \( f \) is \( \mathcal{L} \)-measurable and either (a) \( f \geq 0 \) or (b) \( f \in L^1(\lambda) \), prove that \( f_x \) is \( \mathcal{N} \)-measurable for a.e. \( x \), \( f^y \) is \( \mathcal{M} \)-measurable for a.e. \( y \), and in case (b) \( f_x \) and \( f^y \) are also integrable for a.e. \( x \) and \( y \). Moreover, \( x \mapsto \int f_x \, d\nu, \ y \mapsto \int f^y \, d\mu \) are measurable, and in case (b) they are integrable and you can do the iterated integral in either way. \text{SKIP}

Note that Tonelli and Fubini are often used in conjunction with each other in the following way: first one examines \( \int |f| \, d\mu \times \nu \) by using Tonelli to take iterated integrals. If you show that this value is \( < \infty \), then \( f \in L^1 \) and you can then use Fubini to actually evaluate the integral. Here are some examples of such problems.

2.57. Prove that if \( f \) is Lebesgue integrable on \((0, a)\) and \( g(x) = \int_x^a t^{-1} f(t) \, dt \), then \( g \) is integrable on \((0, a)\) and \( \int_0^a g(x) \, dx = \int_0^a f(x) \, dx \). \text{(Hint:)}

2.58. Let \( f, g \in L^1(m) \). Prove that \( (f * g)(x) = \int f(x - y) g(y) \, dm(y) \) is defined for a.e. \( x \in \mathbb{R}, \ f * g \in L^1(m), \) and that \( \|f * g\|_1 \leq \|f\|_1 \|g\|_1 \). \text{(Hint:)}

2.16. \text{(Hint:)}

2.6. \textbf{The } n\text{-dimensional Lebesgue Integral.}

2.59. We define \( m^n \), the Lebesgue measure on \( \mathbb{R}^n \), as the completion of the \( n \)-fold product of Lebesgue measure on \( \mathbb{R} \) with itself. Its domain will be denoted by \( \mathcal{L}^n \). We shall often just write \( m \) for \( m^n \) and call the measure simply the Lebesgue measure. If \( E = \prod E_j \) is a rectangle in \( \mathbb{R}^n \), we will say that \( E_j \) is a side of \( E \).

Suppose \( E \in \mathcal{L}^n \).

2.59.1. Prove that \( m(E) = \inf\{m(U) : U \supset E, \ U \text{ open} \} = \sup\{m(K) : K \subseteq E, \ K \text{ compact} \} \). \text{(Hint: 1.30.)}
It's not obvious to me how to do the previous procedure so here's an algorithm. Let $R(i)_{j}$ denote $T_{j}$ after replacing the first $i$ sides and let $R(i)_{k,j}$ denote the $k$th side of this rectangle. By 1.30, one can find an open set $U(i)_{k,j}$ such that $R(i)_{k,j} \supset U(i)_{k,j}$ such that the difference in measure is at most $\epsilon = 2^{-i-j} \prod_{k \neq i+1} m(R(i)_{k,j})$, so when one replaces the side with this open set one increases the measure by at most $\epsilon = 2^{-i-j} \prod_{k \neq i+1} m(R(i)_{k,j})$, so one can do this for every side to get an open set $U_{j} \supset T_{j}$ with $m(U_{j}) \leq m(R_{j}) + \epsilon = 2^{-j}$, then do this for every rectangle.

2.59.2. Prove that $E = A_{1} \cup N_{1} = A_{2} \setminus N_{2}$ where $A_{1}$ is an $F_{\sigma}$ set (countable union of closed sets), $A_{2}$ is a $G_{\delta}$ set (countable intersections of open sets), and $m(N_{1}) = m(N_{2}) = 0$.

2.59.3. If $m(E) < \infty$, for any $\epsilon > 0$ there is a finite collection $\{R_{j}\}_{1}^{N}$ of disjoint rectangles whose sides are intervals such that $m(E \Delta \bigcup R_{j}) < \epsilon$. (Hint:)

2.60. Let $f \in L^{1}(m)$, $\epsilon > 0$. Prove that there is a simple function $\phi = \sum a_{j} \chi_{R_{j}}$ where each $R_{j}$ is a product of intervals such that $\int |f - \phi| < \epsilon$, and there is a continuous function $g$ vanishing outside a bounded set such that $\int |f - g| < \epsilon$. (Hint:)

2.59.2.2.32.)

2.61. Skipping a bunch of stuff that I’ve commented out. Highlights are:

- $m$ is translation and rotation invariant.
- If $T \in GL(n, \mathbb{R})$ then $\int f = |\det T| \int f \circ T$ and $m(T(E)) = |\det T| m(E)$.
- Approximating sets by cubes.
- Stuff about diffeomorphisms.

2.7. Integration in Polar Coordinates. SKIP

3. Signed Measures and Differentiation

3.1. Signed Measures.

3.1. A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that $\nu(\emptyset) = 0$, $\nu$ assumes at most one of the values $\pm \infty$, and if $\{E_{j}\}$ is a
sequence of disjoint sets in \( \mathcal{M} \) then \( \nu(\bigcup E_j) = \sum \nu(E_j) \), with the latter sum converging absolutely if \( \sum \nu(E_j) \) is finite.

The absolute convergence condition is only used to make sure the sum is well defined (otherwise by Riemann rearrangement we could have \( \nu(E) \) be any value we wanted) and not used at all when proving any of the results in this section.

All measures are signed measures, and for emphasis we shall sometimes refer to measures as positive measures.

3.1.1. Prove that if \( \mu_1, \mu_2 \) are measures on \( \mathcal{M} \) where at least one of them is finite, then \( \nu := \mu_1 - \mu_2 \) is a signed measure. \( \text{(Hint: } \text{)} \)

3.1.2. Prove that if \( \mu \) is a measure on \( \mathcal{M} \) and \( f : X \to [-\infty, \infty] \) is a measurable function such that at least one of \( \int f^+ \, d\mu, \int f^- \, d\nu \) is finite (in which case we will call \( f \) an extended \( \mu \)-integrable function), then \( \nu(E) := \int_E f \, d\mu \) is a signed measure.

Later in this section we will see that the above two are the only examples of signed measures. I.e. every signed measure can be written as the difference of two positive measures (3.5.) and/or as the integral with respect to some function \( f \in L^1(\mu) \) (3.8.).

3.2. Let \( \nu \) be a signed measure on \( (X, \mathcal{M}) \). If \( \{E_j\} \) is an increasing sequence in \( \mathcal{M} \), prove that \( \nu(\bigcup E_j) = \lim \nu(E_j) \). If \( \{E_j\} \) is a decreasing sequence in \( \mathcal{M} \) and \( |\nu(E_1)| < \infty \), prove that \( \nu(\bigcap E_j) = \lim \nu(E_j) \). \( \text{(Hint: } \text{)} \)

3.3. We will say that a set \( E \in \mathcal{M} \) is positive for \( \nu \) if \( \nu(F) \geq 0 \) for all \( F \in \mathcal{M} \) with \( F \subseteq E \) (we similarly define negative and null sets).

Prove that any measurable subset of a positive set is positive and that the union of any countable family of positive sets is positive. \( \text{(Hint: } \text{)} \)

3.4. [The Hahn Decomposition Theorem] If \( \nu \) is a signed measure on \( (X, \mathcal{M}) \), then there exists a positive set \( P \) and a negative set \( N \) for \( \nu \) such that \( P \cup N = X \) and \( P \cap N = \emptyset \). If \( P', N' \) is another such pair, then \( P \Delta P' = N \Delta N' \) is null for \( \nu \).

3.4.1. Assume wlog that \( \nu \) doesn’t take on the value \(+\infty\). Find a positive set \( P \) such that \( \nu(P) = m = \sup \{\nu(E) : E \text{ positive} \} \), and observe that \( m \) is finite. \( \text{(Hint: } \text{)} \)

3.3. \( \text{ } \) (Hint: 3.2.)
3.4.2. Let $N = X \setminus P$. Prove that $N$ contains no positive sets that aren’t null sets. (Hint: )

3.4.3. Prove that $N$ is negative. (Hint: )

3.4.4. Prove the uniqueness claim. (Hint: )

The decomposition $X = P \sqcup N$ is called a Hahn decomposition for $\nu$.

3.5. We say that two signed measures $\mu, \nu$ on $(X, \mathcal{M})$ are mutually singular, and denote this by $\mu \perp \nu$, if there exists $E, F \in \mathcal{M}$ such that $E \cap F = \emptyset$, $E \cup F = X$, $E$ is null for $\mu$ and $F$ is null for $\nu$. Informally this means that $\mu, \nu$ “live on disjoint sets.” In fact, the notation $\perp$ is meant to convey that $\mu$ lives on the left of the space and $\nu$ the right and they’re totally separated.

[The Jordan Decomposition Theorem] If $\nu$ is a signed measure, prove that there exists unique positive measures $\nu^+, \nu^-$ such that $\nu = \nu^+ - \nu^-$. (Hint: )

3.6. $\nu^+, \nu^-$ are called the positive and negative variations of $\nu$ and $\nu = \nu^+ - \nu^-$ is called the Jordan decomposition of $\nu$. We define the total variation of $\nu$ to be the measure defined by $|\nu| = \nu^+ + \nu^-$. Prove that $E \in \mathcal{M}$ is $\nu$-null iff $|\nu|(E) = 0$, and that $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

3.7. Prove that if $\nu = \lambda - \mu$ with $\lambda$, $\mu$ positive measures, then $\lambda \geq \nu^+$, $\mu \geq \nu^-$ (thus the Jordan decomposition is also the unique “smallest” decomposition). (Hint: )

3.8. Observe that $\nu(E) = \int_E f \, d\mu$ where $\mu = |\nu|$ and $f = \chi_P - \chi_N$, where $P \cup N$ is a Hahn decomposition for $\nu$. 


Some last minute things. We can integrate with respect to a sign measure by defining $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$ and $\int f d\nu = \int f d\nu^+ - \int f d\nu^-$. Lastly, a signed measure $\nu$ is called $(\sigma)$-finite if $|\nu|$ is $(\sigma)$-finite.

3.2. The Lebesgue-Radon-Nikodym Theorem.

3.9. If $\nu$ is a signed measure and $\mu$ a positive measure, we say that $\nu$ is absolutely continuous with respect to $\mu$ and write $\nu \ll \mu$ if $\nu(E) = 0$ for all $E \in \mathcal{M}$ such that $\mu(E) = 0$.

3.9.1. Prove that $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

3.9.2. Prove that $\nu \ll \mu$ iff the null sets of $\mu$ are null sets of $\nu$.

3.9.3. Prove that if $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$. Thus one could see mutual singularity as the antithesis of absolute continuity.

3.10. Let $\nu$ be a finite signed measure and $\mu$ a positive measure on $(X,\mathcal{M})$. Prove that $\nu \ll \mu$ iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\nu(E)| < \epsilon$ whenever $\mu(E) < \delta$. (Hint: ) (Hint: )

3.10.1. Conclude that if $f \in L^1(\mu)$, then for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\int_E f d\mu| < \epsilon$ whenever $\mu(E) < \delta$.

3.11. When $\nu(E) = \int_E f d\mu$ we write $d\nu = f d\mu$ (this notation takes a while to get used to).

[The Lebesgue-Radon-Nikodym Theorem] Let $\nu$ be a $\sigma$-finite signed measure and $\mu$ a $\sigma$-finite positive measure on $(X,\mathcal{M})$. There exists unique $\sigma$-finite signed measures $\lambda, \rho$ on $(X,\mathcal{M})$ such that

$$\nu = \lambda + \rho, \lambda \perp \mu, \rho \ll \mu.$$ Moreover, there is an extended $\mu$-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f d\mu$, and any two such functions are equal $\mu$-a.e.

3.11.1. Suppose that $\nu, \mu$ are finite positive measures and let

$$\mathcal{F} = \{f : X \to [0,\infty] : \int_E f d\mu \leq \nu(E) \forall E \in \mathcal{M}\}.$$ Prove that if $f, g \in \mathcal{F}$, then $\max(f, g) \in \mathcal{F}$. (Hint: )

3.11.2. Note that $\mathcal{F}$ is non-empty since $0 \in \mathcal{F}$. Thus we can define $a = \sup\{f \int f d\mu \in \mathcal{F}\}$, noting that $a \leq \nu(X) < \infty$. Prove that there exists an $f \in \mathcal{F}$ with $\int f d\mu = a$. In particular, $f < \infty$ a.e. so we may take $f$ to be real-valued everywhere. (Hint: ) (Hint: )
3.12.3. Conclude that if \( \nu \ll \mu \), or there exists \( \epsilon > 0 \) and \( E \in \mathcal{M} \) such that \( \mu(E) > 0 \) and \( \nu \geq \epsilon \mu \) on \( E \) (i.e. \( E \) is a positive set for \( \nu - \epsilon \mu \)). \textbf{TODO} (Hint: )

3.12.4. Note that the measure \( d\lambda = d\nu - fd\mu \) is positive since \( f \in \mathcal{F} \). Use the lemma to prove that \( \lambda \) is singular with respect to \( \mu \). We’ll then take \( dp = fd\mu \) to get our decomposition. (Hint: )

3.12.5. Prove that this decomposition is unique. (Hint: 3.9.3.)

3.12.6. Prove the statement for the case \( \mu, \nu \) are \( \sigma \)-finite positive measures. (Hint: ) (Hint: )

3.12.7. Prove the statement for \( \nu \) a signed measure. (Hint: )

3.12. The decomposition from the above theorem is called the Lebesgue decomposition of \( \nu \) with respect to \( \mu \). When \( \nu \ll \mu \) we have \( d\nu = fd\mu \), and in this case we shall denote \( f \) by \( \frac{d\nu}{d\mu} \), the Radon-Nikodym derivative of \( \nu \) with respect to \( \mu \) (again, notation takes a while to get used to).

Suppose that \( \nu \) is a \( \sigma \)-finite signed measure and \( \mu, \lambda \) are \( \sigma \)-finite measures on \( (X, \mathcal{M}) \) such that \( \nu \ll \mu \) and \( \mu \ll \lambda \).

3.12.1. If \( g \in L^1(\nu) \), prove that \( g(d\nu/d\mu) \in L^1(\mu) \) and \( \int gd\nu = \int g\frac{d\nu}{d\mu}d\mu \). (Hint: ) (Hint: )

Note that the “equation” \( d\nu = \frac{d\nu}{d\mu}d\mu \) is the entire reason we use all this (seemingly) strange notation.

3.12.2. Observe that \( \nu \ll \lambda \) and prove that \( \frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda} \) \( \lambda \)-a.e. (Hint: 3.12.1. 2.28.)

3.12.3. Conclude that if \( \mu \ll \lambda \), \( \lambda \ll \mu \) then \( (d\lambda/d\mu)(d\mu/d\lambda) = 1 \) a.e. (with respect to either \( \lambda \) or \( \mu \)).
3.13. The condition that $\mathcal{M}$ be $\sigma$-finite in 3.11. is necessary. Indeed, consider $X = [0, 1]$, $m$ the Lebesgue measure and $\mu$ the counting measure. Prove that $m \ll \mu$ but that there exists no $f$ such that $dm = f d\mu$.

3.14. If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$, observe that there exists a measure $\mu$ such that $\mu_j \ll \mu$ for all $j$, namely $\mu = \sum \mu_j$.

3.15. Let $\mu$ be a Borel measure on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |f| d\mu \leq \|f\|_{L^1(\mathbb{R}^n)}$ for $f$ every bounded Borel measurable function. Prove that $\mu \ll m$ and $d\mu/dm(x) \leq 1$ for $m$-a.e. $x \in \mathbb{R}^n$. (Hint: 3.11.) (Hint: )

3.3. Complex Measures.

3.16. A complex measure on a measurable space $(X, \mathcal{M})$ is a map $\nu : \mathcal{M} \rightarrow \mathbb{C}$ such that (1) $\nu(\emptyset) = 0$ and (2) if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu(\bigcup_{j=1}^\infty E_j) = \sum_{j=1}^\infty \nu(E_j)$, where the series always converges absolutely. In particular, infinite values are not allowed.

Prove that a positive measure is a complex measure iff it is finite. If $\mu$ is a positive measure and $f \in L^1(\mu)$, prove that $fd\mu$ is a complex measure.

3.17. If $\nu$ is a complex measure, we write $\nu_r, \nu_i$ for the real/imaginary parts of $\nu$. The notions developed for signed measures generalize to complex measures.

We define $L^1(\nu) = L^1(\nu_r) \cap L^1(\nu_i)$ and set $\int f d\nu = \int f d\nu_r + i \int f d\nu_i$. If $\nu, \mu$ are complex measures we say that $\nu \perp \mu$ if $\nu_a \perp \mu_b$ for all $a, b = r, i$, and if $\lambda$ is a positive measure we say that $\nu \ll \lambda$ if $\nu_r, \nu_i \ll \lambda$. Most of the results from the previous section apply to complex measures by applying the results to the real and imaginary parts of your complex measure. In particular, verify that we have the following result.

If $\nu$ is a complex measure and $\nu$ is a $\sigma$-finite positive measure on $(X, \mathcal{M})$, there exists a complex measure $\lambda$ and an $f \in L^1(\mu)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$. If also $\lambda' \perp \mu$ and $d\nu = d\lambda' + f' d\mu$, then $\lambda = \lambda'$ and $f = f'\mu$-a.e. We denote this $f$ by $dv/d\mu$.

3.18. For a complex measure $\nu$ we wish to define a positive measure $|\nu|$, the total variation of $\nu$.

3.18.1. Prove that every complex measure satisfies $d\nu = fd\mu$ for some positive measure $\mu$ and $f \in L^1(\mu)$. (Hint: )

3.18.2. If $d\nu = f_1 d\mu_1 = f_2 d\mu_2$ with $\mu_1, \mu_2$ positive measures, prove that $|f_1|d\mu_1 = |f_2|d\mu_2$. (Hint: ) (Hint: )

3.18.3. Conclude that $d|\nu| := |f|d\mu$ where $d\nu = fd\mu$ is well-defined for all complex measures $\nu$. 
3.18.4. Prove that this notation agrees with the earlier notation when \( \nu \) is a signed measure. \( \text{(Hint: } \) \\

3.19. Let \( \nu \) be a complex measure on \((X, \mathcal{M})\).

3.19.1. Prove that \( |\nu(E)| \leq |\nu|(E) \) for all \( E \in \mathcal{M} \).

3.19.2. Prove that \( \nu \ll |\nu| \) and that \( d\nu/d|\nu| \) has absolute value 1 \(|\nu|\)-a.e. \( \text{TODO} \)

3.19.3. Prove that \( L^1(\nu) = L^1(|\nu|) \), and if \( f \in L^1(\nu) \), then \( |\int f\,d\nu| \leq \int |f|d|\nu| \). \( \text{Hint: } 3.19.1. \)

3.20. If \( \nu_1, \nu_2 \) are complex measures on \((X, \mathcal{M})\), prove that \(|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|\). \( \text{Hint: } 3.14. \)

3.4. Differentiation on Euclidean Space. The Radon-Nikodym theorem gives an abstract notion of a “derivative” of a signed or complex measure \( \nu \) with respect to a measure \( \mu \). By restricting our attention to the case \((X, \mathcal{M}, \mu) = (\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, m)\), we’ll be able to define a pointwise derivative of \( \nu \). Namely, let \( B(r, x) \) be the open ball of radius \( r \) about \( x \). Then one can consider the limit

\[
F(x) = \lim_{r \to 0} \frac{\nu(B(r, x))}{m(B(r, x))},
\]

whenever this quantity exists. If \( \nu \ll m \) so that \( d\nu = f\,dm \), then \( \nu(B(r, x))/m(B(r, x)) \) is simply the average value of \( f \) on \( B(r, x) \), so one would hope that \( F = f \) \( m \)-a.e., and this turns out to be the case provided \( \nu(B(r, x)) \) is finite for all \( r, x \). This can be seen as a sort of generalization of the fundamental theorem of calculus: the derivative of the indefinite integral of \( f \) (namely, \( \nu \)), is \( f \).

For the rest of the section “integrable” and “almost everywhere” will refer to the Lebesgue measure.

3.21. Let \( \mathcal{C} \) be a collection of open balls in \( \mathbb{R}^n \) and let \( U = \bigcup_{B \in \mathcal{C}} B \). If \( c < m(u) \), prove that there exists disjoint \( B_1, \ldots, B_k \in \mathcal{C} \) such that \( \sum_{j} m(B_j) > 3^{-n}c \).

You can assume that \( m(B(r, x)) = c r^n \) for some \( c = m(B(1, 0)) \), which is proven in section 2.7. \( \text{Hint: } 1.30. \)

3.22. A measurable function \( f : \mathbb{R}^n \to \mathbb{C} \) is called locally integrable (with respect to the Lebesgue measure) if \( \int_K |f(x)|\,dx < \infty \) for every bounded measurable set \( K \subseteq \mathbb{R}^n \). We denote the space of locally integrable functions by \( L^1_{\text{loc}} \). If \( f \in L^1_{\text{loc}}, x \in \mathbb{R}^n \) and \( r > 0 \), we define \( A_r f(x) \) to be the average value of
3.23. If \( f \in L^1_{\text{loc}} \) we define its Hardy-Littlewood maximal function \( Hf \) by \( Hf(x) = \sup_{r>0} A_r[f](x) \). Prove that \( Hf \) is measurable. \textbf{(Hint: 3.22.)}

3.24. [The Maximal Theorem] Prove that there is a constant \( C > 0 \) such that for all \( f \in L^1 \) and all \( \alpha > 0 \),

\[
m\{|x : Hf(x) > \alpha\} \leq C/\alpha \int |f(x)|dx.
\]

\textbf{(Hint: 3.21.)} \textbf{(Hint: 3.21.)}

3.24.1. Prove that if \( f \in L^1(\mathbb{R}^n), \ f \neq 0 \), then there exists \( C, R > 0 \) such that \( Hf(x) \geq C|x|^{-n} \) for \( |x| > R \). Conclude that \( m\{|x : Hf(x) > \alpha\} \geq C'/\alpha \) when \( \alpha \) is small, so the estimate in the maximal theorem is essentially sharp. \textbf{TODO(Hint: 3.21.)}

3.25. The following three results are successively sharper versions of the fundamental differentiation theorem.

If \( f \in L^1_{\text{loc}} \) prove that \( \lim_{r \to 0} A_r f(x) = f(x) \) for a.e. \( x \in \mathbb{R}^n \). \textbf{TODO(Hint: 3.21.)}

3.26. Can be rephrased as: if \( f \in L^1_{\text{loc}} \), then

\[
\lim_{r \to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)|dy = 0 \text{ a.e.}
\]

We can actually get a stronger statement where we replace the integrand with its absolute value. Namely, define the Lebesgue set \( L_f \) of \( f \) by

\[
L_f = \{x : \lim_{r \to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y) - f(x)|dy = 0\}.
\]
Prove that if \( f \in L^1_{\text{loc}} \), then \( m((L_f)^c) = 0 \). TODO

3.27. A family \( \{E_r \}_{r > 0} \) of Borel subsets of \( \mathbb{R}^n \) is said to shrink nicely to \( x \in \mathbb{R}^n \) if (1) \( E_r \subseteq B(r, x) \) for each \( r \), (2) there is a constant \( \alpha > 0 \), independent of \( r \), such that \( m(E_r) > \alpha m(B(r, x)) \). Note that the \( E_r \) need not contain \( x \) itself.

[The Lebesgue Differentiation Theorem] Suppose \( f \in L^1_{\text{loc}} \). Prove that for every \( x \) in the Lebesgue set of \( f \) (in particular, for a.e. \( x \)) that we have

\[
\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0, \quad \lim_{r \to \infty} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x).
\]

(Hint: 3.26.)

3.28. A Borel measure \( \nu \) on \( \mathbb{R}^n \) is said to be regular if (1) \( \nu(K) < \infty \) for every compact \( K \) and (2) \( \nu(E) = \inf\{\nu(U) : U \text{ open}, E \subseteq U\} \) for every \( E \in \mathcal{B}_{\mathbb{R}^n} \).

Actually, (2) is implied by (1) (for \( n = 1 \) follows from 1.28. and 1.30. and we’ll prove it for arbitrary \( n \) in 7.2), but for now we’ll assume (2) explicitly.

A signed or complex Borel measure \( \nu \) will be called regular if \( |\nu| \) is regular. Note that all regular measures are \( \sigma \)-finite. Prove that if \( f \in L^1(\mathbb{R}^n) \), then the measure \( f dm \) is regular iff \( f \in L^1_{\text{loc}} \). TODO

3.29. TODO Let \( \nu \) be a regular signed or complex Borel measure on \( \mathbb{R}^n \) and let \( d\nu = d\lambda + f dm \) be its Lebesgue-Radon-Nikodym representation. Prove that for \( m \)-almost every \( x \in \mathbb{R}^n \)

\[
\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x),
\]

for every family \( \{E_r\}_{r > 0} \) that shrinks nicely to \( x \). TODO

3.5. Functions of Bounded Variation.

3.30. TODO

4. General Techniques

TODO Possibly include references to a bunch of examples using each technique.

4.1. Prove statements for simple cases, then keep building on these results until you get the whole thing. E.g. prove a statement for characteristic functions, then simple functions, then \( L^+ \) functions, then \( L^1 \) functions. E.g. Prove a statement for \( \mu, \nu \) finite positive measures, then \( \sigma \)-finite, then signed measures.

4.2. When \( \mu(X) < \infty \), often your argument involves splitting \( X \) into a bad part (whose measure goes to 0 faster than the “measure of badness”) and a good part (whose measure of badness goes to 0 at any rate, which is fine because the good part has measure at most \( \mu(X) < \infty \)).

4.3. Use approximations (characteristic functions, continuous functions, suitable subsequences).
4.4. Take liminfs/limsups.

4.5. Take $E_\alpha$ corresponding to something like $1/n^2$, then use countability of unions and or/summability of those values.