The $\mathcal{F}$-saturation game

Let $\mathcal{F}$ be a set of (forbidden graphs). We consider the following two player zero sum game played by Max and Mini.
The $\mathcal{F}$-saturation game

Let $\mathcal{F}$ be a set of (forbidden graphs). We consider the following two player zero sum game played by Max and Mini.

The game starts with an initially empty graph $G$ on $n$ vertices. Max and Mini alternate turns adding a new edge to $G$, with the only restriction being that neither play can add an edge that would create some $F \in \mathcal{F}$ as a subgraph in $G$. Thus Max wants the game to last as long as possible, while Mini wants the game to end as quickly as possible.
The $\mathcal{F}$-saturation game

Let $\mathcal{F}$ be a set of (forbidden graphs). We consider the following two player zero sum game played by Max and Mini.

The game starts with an initially empty graph $G$ on $n$ vertices. Max and Mini alternate turns adding a new edge to $G$, with the only restriction being that neither play can add an edge that would create some $F \in \mathcal{F}$ as a subgraph in $G$. The game ends when no more edges can be added to $G$ (that is, when $G$ is $\mathcal{F}$-saturated).
The $\mathcal{F}$-saturation game

Let $\mathcal{F}$ be a set of (forbidden graphs). We consider the following two player zero sum game played by Max and Mini.

The game starts with an initially empty graph $G$ on $n$ vertices. Max and Mini alternate turns adding a new edge to $G$, with the only restriction being that neither play can add an edge that would create some $F \in \mathcal{F}$ as a subgraph in $G$. The game ends when no more edges can be added to $G$ (that is, when $G$ is $\mathcal{F}$-saturated).

When the game ends, Max gets a point for every edge in $G$ at the end of the game and Mini loses a point for every edge in $G$. 
The $\mathcal{F}$-saturation game

Let $\mathcal{F}$ be a set of (forbidden graphs). We consider the following two player zero sum game played by Max and Mini.

The game starts with an initially empty graph $G$ on $n$ vertices. Max and Mini alternate turns adding a new edge to $G$, with the only restriction being that neither play can add an edge that would create some $F \in \mathcal{F}$ as a subgraph in $G$. The game ends when no more edges can be added to $G$ (that is, when $G$ is $\mathcal{F}$-saturated).

When the game ends, Max gets a point for every edge in $G$ at the end of the game and Mini loses a point for every edge in $G$. Thus Max wants the game to last as long as possible, while Mini wants the game to end as quickly as possible.
The $\mathcal{F}$-saturation game

Example: the $\{C_3\}$-saturation game. Max goes first.
The $\mathcal{F}$-saturation game

Example: the $\{C_3\}$-saturation game. Max goes first.
Example: the \( \{C_3\} \)-saturation game. Max goes first.
The $\mathcal{F}$-saturation game

Example: the $\left\{ C_3 \right\}$-saturation game. Max goes first.
Example: the \( \{C_3\} \)-saturation game. Max goes first. The graph is now \( C_3 \)-saturated so the game ends. Max gets 4 points (the best he could possibly do) and Mini loses 4 points.
The $\mathcal{F}$-saturation game

Example: the $\{C_3\}$-saturation game. Max goes first.

The graph is now $C_3$-saturated so the game ends. Max gets 4 points (the best he could possibly do) and Mini loses 4 points.
Let $\text{sat}_g(\mathcal{F}; n)$ denote the number of edges in $G$ at the end of the $\mathcal{F}$-saturation game when both players play optimally. The goal is to find this value, which is known as the $\mathcal{F}$-game saturation number.
Let $s\text{at}_g(F; n)$ denote the number of edges in $G$ at the end of the $F$-saturation game when both players play optimally. The goal is to find this value, which is known as the $F$-game saturation number.

**Example**

$$n - 1 \leq s\text{at}_g(\{C_3\}; n) \leq \left\lfloor \frac{1}{4}n^2 \right\rfloor.$$
The \( \mathcal{F} \)-game saturation number

**Theorem (Furedi-Reimer-Sersess, 1992)**

\[
\text{sat}_g(\{ C_3 \}; n) \geq \frac{1}{2} n \log n + o(n \log n).
\]
The $F$-game saturation number

Theorem (Furedi-Reimer-Sersess, 1992)

$$\text{sat}_g(\{C_3\}; n) \geq \frac{1}{2} n \log n + o(n \log n).$$

Theorem (Biró-Horn-Wildstrom, 2014)

$$\text{sat}_g(\{C_3\}; n) \leq \frac{26}{121} n^2 + o(n^2).$$
The \( F \)-game saturation number

**Theorem (Furedi-Reimer-Sersess, 1992)**

\[
\text{sat}_g(\{C_3\}; n) \geq \frac{1}{2} n \log n + o(n \log n).
\]

**Theorem (Biró-Horn-Wildstrom, 2014)**

\[
\text{sat}_g(\{C_3\}; n) \leq \frac{26}{121} n^2 + o(n^2).
\]

These are the only known bounds for the triangle-free game.
The $\mathcal{F}$-game saturation number

**Theorem (Furedi-Reimer-Sersess, 1992)**

$$\text{sat}_g(\{C_3\}; n) \geq \frac{1}{2} n \log n + o(n \log n).$$

**Theorem (Biró-Horn-Wildstrom, 2014)**

$$\text{sat}_g(\{C_3\}; n) \leq \frac{26}{121} n^2 + o(n^2).$$

These are the only known bounds for the triangle-free game. Our goal is to establish a lower bound for a related game, namely the $\{C_3, C_5\}$-saturation game.
The $\mathcal{F}$-game saturation number

**Theorem (Furedi-Reimer-Sersess, 1992)**

$$\text{sat}_g(\{C_3\}; n) \geq \frac{1}{2} n \log n + o(n \log n).$$

**Theorem (Biró-Horn-Wildstrom, 2014)**

$$\text{sat}_g(\{C_3\}; n) \leq \frac{26}{121} n^2 + o(n^2).$$

These are the only known bounds for the triangle-free game. Our goal is to establish a lower bound for a related game, namely the $\{C_3, C_5\}$-saturation game. Key idea: Max can force the graph to be bipartite throughout this game.
The $\{C_3, C_5\}$-saturation game

In general, let $X^t$ denote $X$ after $t$ edges have been added in the game, e.g. $G^t$ denotes the graph after $t$ edges have been played, $e^t$ denotes the edge added at time $t$, etc.

Max wishes to end each of his turns such that $G^t$ satisfies the following conditions.

1. $G^t$ contains exactly one non-trivial connected component, and this component is bipartite with bipartition $U^t \cup V^t$.

   Identify two adjacent vertices $u \in U^t, v \in V^t$. Let $U^t_b = U^t \setminus N(v)$ and $V^t_b = V^t \setminus N(u)$ (the bad vertices).

2. Every vertex of $U^t \cup V^t$ is adjacent to a vertex in $N(u) \cup N(v)$. How can Max play so that he can achieve this?
In general, let $X^t$ denote $X$ after $t$ edges have been added in the game, e.g. $G^t$ denotes the graph after $t$ edges have been played, $e^t$ denotes the edge added at time $t$, etc. Max wishes to end each of his turns such that $G^t$ satisfies the following conditions.

1. $G^t$ contains exactly one non-trivial connected component, and this component is bipartite with bipartition $U^t \cup V^t$.

2. Every vertex of $U^t \cup V^t$ is adjacent to a vertex in $N(u^t) \cup N(v^t)$ (the bad vertices).
The \( \{C_3, C_5\} \)-saturation game

In general, let \( X^t \) denote \( X \) after \( t \) edges have been added in the game, e.g. \( G^t \) denotes the graph after \( t \) edges have been played, \( e^t \) denotes the edge added at time \( t \), etc. Max wishes to end each of his turns such that \( G^t \) satisfies the following conditions.

\[ (1*) \quad G^t \text{ contains exactly one non-trivial connected component, and this component is bipartite with bipartition } U^t \cup V^t. \]

Identify two adjacent vertices \( u \in U^t \), \( v \in V^t \). Let \( U^t_{b} = U^t \setminus N(v)^t \) and \( V^t_{b} = V^t \setminus N(u)^t \) (the bad vertices).

\[ (2*) \quad \text{Every vertex of } U^t \cup V^t \text{ is adjacent to a vertex in } N(u)^t \cup N(v)^t. \]
In general, let $X^t$ denote $X$ after $t$ edges have been added in the game, e.g. $G^t$ denotes the graph after $t$ edges have been played, $e^t$ denotes the edge added at time $t$, etc. Max wishes to end each of his turns such that $G^t$ satisfies the following conditions.

(1*) $G^t$ contains exactly one non-trivial connected component, and this component is bipartite with bipartition $U^t \cup V^t$.

Identify two adjacent vertices $u \in U^t$, $v \in V^t$. Let $U^t_b = U^t \setminus N(v)^t$ and $V^t_b = V^t \setminus N(u)^t$ (the bad vertices).
The \( \{C_3, C_5\}\)-saturation game

In general, let \( X^t \) denote \( X \) after \( t \) edges have been added in the game, e.g. \( G^t \) denotes the graph after \( t \) edges have been played, \( e^t \) denotes the edge added at time \( t \), etc. Max wishes to end each of his turns such that \( G^t \) satisfies the following conditions.

\((1^*)\) \( G^t \) contains exactly one non-trivial connected component, and this component is bipartite with bipartition \( U^t \cup V^t \).

Identify two adjacent vertices \( u \in U^t, v \in V^t \). Let \( U^t_b = U^t \setminus N(v)^t \) and \( V^t_b = V^t \setminus N(u)^t \) (the bad vertices).

\((2^*)\) Every vertex of \( U^t \cup V^t \) is adjacent to a vertex in \( N(u)^t \cup N(v)^t \).
The $\{C_3, C_5\}$-saturation game

In general, let $X^t$ denote $X$ after $t$ edges have been added in the game, e.g. $G^t$ denotes the graph after $t$ edges have been played, $e^t$ denotes the edge added at time $t$, etc. Max wishes to end each of his turns such that $G^t$ satisfies the following conditions.

(1*) $G^t$ contains exactly one non-trivial connected component, and this component is bipartite with bipartition $U^t \cup V^t$.

Identify two adjacent vertices $u \in U^t$, $v \in V^t$. Let $U^t_b = U^t \setminus N(v)^t$ and $V^t_b = V^t \setminus N(u)^t$ (the bad vertices).

(2*) Every vertex of $U^t \cup V^t$ is adjacent to a vertex in $N(u)^t \cup N(v)^t$.

How can Max play so that he can achieve this?
The $\{C_3, C_5\}$-saturation game

Inductively assume that Max plays so $G^{t-2}$ satisfies (1*) and (2*).
The $\{C_3, C_5\}$-saturation game

Inductively assume that Max plays so $G^{t-2}$ satisfies (1*) and (2*). What if $e^{t-1} = u'v'$, $u' \in U^{t-2}$, $v' \in V^{t-2}$ (an Internal move)?
The \( \{C_3, C_5\} \)-saturation game

Inductively assume that Max plays so \( G^{t-2} \) satisfies (1*) and (2*). What if \( e^{t-1} = u'v' \), \( u' \in U^{t-2} \), \( v' \in V^{t-2} \) (an Internal move)?
The $\{C_3, C_5\}$-saturation game

Inductively assume that Max plays so $G^{t-2}$ satisfies (1*) and (2*). What if $e^{t-1} = xy$, $x, y \notin U^{t-2} \cup V^{t-2}$ (an Outside move)?
The \( \{C_3, C_5\}\)-saturation game

Inductively assume that Max plays so \( G^{t-2} \) satisfies (1*) and (2*). What if \( e^{t-1} = xy, \ x, y \notin U^{t-2} \cup V^{t-2} \) (an Outside move)?
The \( \{C_3, C_5\} \)-saturation game

Inductively assume that Max plays so \( G^{t-2} \) satisfies (1*) and (2*). What if \( e^{t-1} = xv', \ x \notin U^{t-2} \cup V^{t-2}, \ v' \in V^{t-2} \) (Add to \( U \)?)

\[
\begin{align*}
N(u)^t & \quad N(v)^t \\
U_b & \quad V_b^t \\
\end{align*}
\]
Inductively assume that Max plays so $G^{t-2}$ satisfies (1*) and (2*). What if $e^{t-1} = xv'$, $x \not\in U^{t-2} \cup V^{t-2}$, $v' \in V^{t-2}$ (Add to $U$)?

The $\{C_3, C_5\}$-saturation game

\begin{itemize}
  \item $u$
  \item $N(v)^t$
  \item $U_b^t$
  \item $v$
  \item $N(u)^t$
  \item $V_b^t$
\end{itemize}
The \( \{C_3, C_5\} \)-saturation game

Inductively assume that Max plays so \( G^{t-2} \) satisfies (1*) and (2*). What if \( e^{t-1} = v'v'', \ v', v'' \in V^{t-2} \)?
The \( \{C_3, C_5\} \)-saturation game

**Lemma**

Let \( t \) be such that \( G^t \) satisfies (1*) and (2*). Then \( U^{t+1} \) and \( V^{t+1} \) are independent sets for any valid choice of \( e^{t+1} \) in the \( \{C_3, C_5\} \)-saturation game for \( k \geq 2 \).
The \( \{C_3, C_5\} \)-saturation game

**Lemma**

Let \( t \) be such that \( G^t \) satisfies (1*) and (2*). Then \( U^{t+1} \) and \( V^{t+1} \) are independent sets for any valid choice of \( e^{t+1} \) in the \( \{C_3, C_5\} \)-saturation game for \( k \geq 2 \).

**Proof.**

\( U^t \) and \( V^t \) are independent sets since \( G^t \) satisfies (1*). Assume \( e^{t+1} = v'v'' \) with \( v', v'' \subseteq V^t \).
The \( \{C_3, C_5\}\)-saturation game

**Lemma**

Let \( t \) be such that \( G^t \) satisfies \((1^*)\) and \((2^*)\). Then \( U^{t+1} \) and \( V^{t+1} \) are independent sets for any valid choice of \( e^{t+1} \) in the \( \{C_3, C_5\}\)-saturation game for \( k \geq 2 \).

**Proof.**

\( U^t \) and \( V^t \) are independent sets since \( G^t \) satisfies \((1^*)\). Assume \( e^{t+1} = v'v'' \) with \( v', v'' \in V^t \). By \((2^*)\) there exists \( u', u'' \in N(v)^t \) that are neighbors of \( v' \) and \( v'' \).
The \{C_3, C_5\}-saturation game

Lemma

Let \( t \) be such that \( G^t \) satisfies (1*) and (2*). Then \( U^{t+1} \) and \( V^{t+1} \) are independent sets for any valid choice of \( e^{t+1} \) in the \{\( C_3, C_5 \)\}-saturation game for \( k \geq 2 \).

Proof.

\( U^t \) and \( V^t \) are independent sets since \( G^t \) satisfies (1*). Assume \( e^{t+1} = v'v'' \) with \( v', v'' \in V^t \). By (2*) there exists \( u', u'' \in N(v)^t \) that are neighbors of \( v' \) and \( v'' \). If \( u' = u'' \), then \( G^{t+1} \) contains the 3-cycle \( v'u'v'' \), otherwise it contains the 5-cycle \( v'u'v'v'' \).
The $\{C_3, C_5\}$-saturation game

Lemma

Let $t$ be such that $G^t$ satisfies (1*) and (2*). Then $U^{t+1}$ and $V^{t+1}$ are independent sets for any valid choice of $e^{t+1}$ in the $\{C_3, C_5\}$-saturation game for $k \geq 2$.

Proof.

$U^t$ and $V^t$ are independent sets since $G^t$ satisfies (1*). Assume $e^{t+1} = v'v''$ with $v', v'' \in V^t$. By (2*) there exists $u', u'' \in N(v)^t$ that are neighbors of $v'$ and $v''$. If $u' = u''$, then $G^{t+1}$ contains the 3-cycle $v'u'v''$, otherwise it contains the 5-cycle $v'u'vu''v''$. These cycles are forbidden, a contradiction.
The \( \{C_3, C_5\}\)-saturation game

**Lemma**

Let \( t \) be such that \( G^t \) satisfies \((1*)\) and \((2*)\). Then \( U^{t+1} \) and \( V^{t+1} \) are independent sets for any valid choice of \( e^{t+1} \) in the \( \{C_3, C_5\}\)-saturation game for \( k \geq 2 \).

**Proof.**

\( U^t \) and \( V^t \) are independent sets since \( G^t \) satisfies \((1*)\). Assume \( e^{t+1} = v'v'' \) with \( v', v'' \in V^t \). By \((2*)\) there exists \( u', u'' \in N(v)^t \) that are neighbors of \( v' \) and \( v'' \). If \( u' = u'' \), then \( G^{t+1} \) contains the 3-cycle \( v'vu''v'' \), otherwise it contains the 5-cycle \( v'vu'vu''v'' \). These cycles are forbidden, a contradiction.

Given this lemma, Mini can only do Internal, Outside, and Add to \( U/V \) moves, so Max can indeed play so that \((1*)\) and \((2*)\) are maintained.
The \( \{C_3, C_5\} \)-saturation game

With this strategy Max can play so that the game stays bipartite, but he can’t control how large the parts are at the end.
The \( \{C_3, C_5\} \)-saturation game

With this strategy Max can play so that the game stays bipartite, but he can’t control how large the parts are at the end. Solution: use a stronger induction.
The \( \{C_3, C_5\}\)-saturation game

With this strategy Max can play so that the game stays bipartite, but he can’t control how large the parts are at the end. Solution: use a stronger induction.

\[
(3^*) \quad b^t_U := \left| V^t_b \right| + (|U^t| - 2|V^t|) \leq 0,
\]
\[
b^t_V := \left| U^t_b \right| + (|V^t| - 2|U^t|) \leq 0.
\]
The \( \{ C_3, C_5 \} \)-saturation game

With this strategy Max can play so that the game stays bipartite, but he can’t control how large the parts are at the end. Solution: use a stronger induction.

\[(3^*) \quad b_U^t := |V_b^t| + (|U^t| - 2|V^t|) \leq 0, \]
\[b_V^t := |U_b^t| + (|V^t| - 2|U^t|) \leq 0. \]

The idea with this property is that \( |U^t| \) and \( |V^t| \) are always within a factor of two of each other.
The $\{C_3, C_5\}$-saturation game

With this strategy Max can play so that the game stays bipartite, but he can’t control how large the parts are at the end. Solution: use a stronger induction.

\[(3^*)\quad b_U^t := |V_b^t| + (|U^t| - 2|V^t|) \leq 0,\]
\[b_V^t := |U_b^t| + (|V^t| - 2|U^t|) \leq 0.\]

The idea with this property is that $|U^t|$ and $|V^t|$ are always within a factor of two of each other. Further, if $|U^t|$ is much larger than $|V^t|$, then there must be few bad $V_b^t$ vertices.
The \( \{C_3, C_5\}\)-saturation game

With this strategy Max can play so that the game stays bipartite, but he can’t control how large the parts are at the end. Solution: use a stronger induction.

\[(3*) \quad b_U^t := |V_b^t| + (|U^t| - 2|V^t|) \leq 0, \quad b_V^t := |U_b^t| + (|V^t| - 2|U^t|) \leq 0.\]

The idea with this property is that \(|U^t|\) and \(|V^t|\) are always within a factor of two of each other. Further, if \(|U^t|\) is much larger than \(|V^t|\), then there must be few bad \(V_b^t\) vertices.

If Mini does an Internal or Outside move then Max acts as he did before, and with this \(b_U^t, b_V^t\) don’t increase.
The \( \{C_3, C_5\}\)-saturation game

With this strategy Max can play so that the game stays bipartite, but he can’t control how large the parts are at the end. Solution: use a stronger induction.

\[
(3*) \quad b^t_U := |V^t_b| + (|U^t| - 2|V^t|) \leq 0,
\]
\[
b^t_V := |U^t_b| + (|V^t| - 2|U^t|) \leq 0.
\]

The idea with this property is that \(|U^t|\) and \(|V^t|\) are always within a factor of two of each other. Further, if \(|U^t|\) is much larger than \(|V^t|\), then there must be few bad \(V^t_b\) vertices.

If Mini does an Internal or Outside move then Max acts as he did before, and with this \(b^t_U, b^t_V\) don’t increase. However, Max has to be more careful when Mini plays an Add to \(U\) move.
The \( \{ C_3, C_5 \} \)-saturation game

Case 1: \(|U^{t+1}| \leq 2|V^{t+1}|\).

\[
\begin{align*}
& b^t_U = |V^t_b| + (|U^t| - 2|V^t|) = 0, \\
& b^t_V = |U^t_b| + (|V^t| - 2|U^t|) = -5.
\end{align*}
\]
The $\{C_3, C_5\}$-saturation game

Case 1: $|U^{t+1}| \leq 2|V^{t+1}|$.

$$b_U^{t+2} = |V_b^{t+2}| + (|U^{t+2}| - 2|V^{t+2}|) = 0,$$
$$b_V^{t+2} = |U_b^{t+2}| + (|V^{t+2}| - 2|U^{t+2}|) = -6.$$
The \( \{C_3, C_5\}\)-saturation game

Case 2: \( |U^{t+1}| > 2|V^{t+1}| \).

\[
\begin{align*}
    b^t_U &= |V^t_b| + (|U^t| - 2|V^t|) = 0, \\
    b^t_V &= |U^t_b| + (|V^t| - 2|U^t|) = -7.
\end{align*}
\]
The $\{C_3, C_5\}$-saturation game

Case 2: $|U^{t+1}| > 2|V^{t+1}|$.

\[ b_U^{t+2} = |V_b^{t+2}| + (|U^{t+2}| - 2|V^{t+2}|) = -1, \]
\[ b_V^{t+2} = |U_b^{t+2}| + (|V^{t+2}| - 2|U^{t+2}|) = -7. \]
The $\{C_3, C_5\}$-saturation game

We conclude that Max can play so that he maintains these conditions (as long as the graph contains isolated vertices).
The $\{C_3, C_5\}$-saturation game

We conclude that Max can play so that he maintains these conditions (as long as the graph contains isolated vertices).

Theorem (S., 2019)

$$\text{sat}_g(\{C_3, C_5\}; n) \geq \frac{2}{9} n^2 + o(n^2).$$
The \( \{C_3, C_5\} \)-saturation game

We conclude that Max can play so that he maintains these conditions (as long as the graph contains isolated vertices).

**Theorem (S., 2019)**

\[
sat_g(\{C_3, C_5\}; n) \geq \frac{2}{9} n^2 + o(n^2).
\]

**Proof.**

Max follows the strategy defined beforehand as long as there exists isolated vertices in \( G^t \), afterwards he plays arbitrarily.
The \( \{C_3, C_5\} \)-saturation game

We conclude that Max can play so that he maintains these conditions (as long as the graph contains isolated vertices).

**Theorem (S., 2019)**

\[
sat_g(\{C_3, C_5\}; n) \geq \frac{2}{9}n^2 + o(n^2).
\]

**Proof.**

Max follows the strategy defined beforehand as long as there exists isolated vertices in \( G^t \), afterwards he plays arbitrarily. At the end of the game, \( G \) will be a complete bipartite graph with, say, \(|V| \leq |U| \leq 2|V| + 1\), and hence contains at least \( \frac{2}{9}n^2 + o(n^2) \) edges.
Improving the constant

We’ve shown that $\text{sat}_g(\{C_3, C_5\}; n)$ is quadratic, but what can be said about the implicit constant?
Improving the constant

We’ve shown that \( sat_g(\{C_3, C_5\}; n) \) is quadratic, but what can be said about the implicit constant?

**Theorem (S., 2019)**

\[ sat_g(\{C_3, C_5\}; n) \geq \frac{6}{25} n^2 + o(n^2). \]
Improving the constant

We’ve shown that $\text{sat}_g(\{C_3, C_5\}; n)$ is quadratic, but what can be said about the implicit constant?

**Theorem (S., 2019)**

$$\text{sat}_g(\{C_3, C_5\}; n) \geq \frac{6}{25} n^2 + o(n^2).$$

Essentially one uses the same strategy as before but with a stronger induction. Namely, Max maintains the following.

(3*) $b_U^t := |V_b^t| + (|U^t| - \frac{3}{2}|V^t| - 2) \leq 0$,

$$b_U^t := |U_b^t| + (|V^t| - \frac{3}{2}|U^t| - 2) \leq 0.$$

(4*) $b_U^t + b_V^t \leq -2$. 
We've shown that \( \text{sat}_g(\{C_3, C_5\}; n) \) is quadratic, but what can be said about the implicit constant?

**Theorem (S., 2019)**

\[
\text{sat}_g(\{C_3, C_5\}; n) \geq \frac{6}{25} n^2 + o(n^2).
\]

Essentially one uses the same strategy as before but with a stronger induction. Namely, Max maintains the following.

\[b^t_U := |V^t_b| + (|U^t| - \frac{3}{2}|V^t| - 2) \leq 0,\]
\[b^t_V := |U^t_b| + (|V^t| - \frac{3}{2}|U^t| - 2) \leq 0.\]

\[(4^*) \quad b^t_U + b^t_V \leq -2.\]

The main idea is that \(4^*\) guarantees that one of \(b^t_U, b^t_V \leq -1\), and hence one of the sets \(U^t, V^t\) can afford to have its structure disrupted.
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

This same proof holds for any set of odd cycles $C$ with $C_3, C_5 \in C$. 

Theorem (S., 2019)

For $k \geq 4$,

$$sg\left(\{C_3, \ldots, C_{2k+1}\}; n\right) \geq \left(\frac{1}{4} - \frac{1}{5^2}k^2\right)n^2 + o\left(n^2\right),$$

$$sg\left(\{C_3, \ldots, C_{2k+1}\}; n\right) \leq \left(\frac{1}{4} - \frac{1}{20^2}k^4\right)n^2 + o\left(n^2\right).$$

Idea for the lower bound: call a vertex bad if it's roughly distance $k$ away from $u$ or $v$ (as opposed to those that simply aren't adjacent to $u/v$). By being more careful in the previous argument, and by making a slight tweak to the strategy, one can replace the $3/2$ we had before with $\gamma_k \rightarrow 1$. 
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

This same proof holds for any set of odd cycles $\mathcal{C}$ with $C_3, C_5 \in \mathcal{C}$. Can Max do better if we forbid larger cycles?
The \( \{C_3, \ldots, C_{2k+1}\} \)-saturation game

This same proof holds for any set of odd cycles \( C \) with \( C_3, C_5 \in C \). Can Max do better if we forbid larger cycles?

**Theorem (S., 2019)**

For \( k \geq 4 \),

\[
\text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \geq \left( \frac{1}{4} - \frac{1}{5k^2} \right) n^2 + o(n^2)
\]
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

This same proof holds for any set of odd cycles $C$ with $C_3, C_5 \in C$. Can Max do better if we forbid larger cycles?

**Theorem (S., 2019)**

*For $k \geq 4$,*

$$\text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \geq \left(\frac{1}{4} - \frac{1}{5k^2}\right)n^2 + o(n^2),$$

$$\text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \leq \left(\frac{1}{4} - \frac{1}{20^6k^4}\right)n^2 + o(n^2).$$
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

This same proof holds for any set of odd cycles $C$ with $C_3, C_5 \in C$. Can Max do better if we forbid larger cycles?

**Theorem (S., 2019)**

For $k \geq 4$,

$$\text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \geq \left( \frac{1}{4} - \frac{1}{5k^2} \right) n^2 + o(n^2),$$

$$\text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \leq \left( \frac{1}{4} - \frac{1}{20^6 k^4} \right) n^2 + o(n^2).$$

Idea for the lower bound: call a vertex bad if it’s roughly distance $k$ away from $u$ or $v$ (as opposed to those that simply aren’t adjacent to $u/v$).
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

This same proof holds for any set of odd cycles $\mathcal{C}$ with $C_3, C_5 \in \mathcal{C}$. Can Max do better if we forbid larger cycles?

**Theorem (S., 2019)**

For $k \geq 4$,

$$\text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \geq \left( \frac{1}{4} - \frac{1}{5k^2} \right) n^2 + o(n^2),$$

$$\text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \leq \left( \frac{1}{4} - \frac{1}{20^6k^4} \right) n^2 + o(n^2).$$

Idea for the lower bound: call a vertex bad if it’s roughly distance $k$ away from $u$ or $v$ (as opposed to those that simply aren’t adjacent to $u/v$). By being more careful in the previous argument, and by making a slight tweak to the strategy, one can replace the $\frac{3}{2}$ we had before with $\gamma_k \to 1$. 
The \( \{C_3, \ldots, C_{2k+1}\} \)-saturation game

The upper bound for \( \text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \) is significantly harder.
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

The upper bound for $\text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n)$ is significantly harder. We’ve shown that Max can guarantee that $G^t$ stays bipartite, so Mini can’t utilize any strategy that requires her to create many odd cycles.
The \( \{C_3, \ldots, C_{2k+1}\} \)-saturation game

The upper bound for \( \text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \) is significantly harder. We’ve shown that Max can guarantee that \( G^t \) stays bipartite, so Mini can’t utilize any strategy that requires her to create many odd cycles. Conversely, one can show that if Mini doesn’t try and create any odd cycles, then Max can play so that \( G^t \) ends with \( \frac{1}{4} n^2 \) edges.
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

The upper bound for $\text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n)$ is significantly harder. We’ve shown that Max can guarantee that $G^t$ stays bipartite, so Mini can’t utilize any strategy that requires her to create many odd cycles. Conversely, one can show that if Mini doesn’t try and create any odd cycles, then Max can play so that $G^t$ ends with $\frac{1}{4} n^2$ edges. Thus any strategy of Mini’s giving a non-trivial bound has to attempt to make odd cycles, while making sure that the final graph ends up unbalanced if Max stops her from doing so.
The \( \{C_3, \ldots, C_{2k+1}\} \)-saturation game

The upper bound for \( \text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \) is significantly harder. We’ve shown that Max can guarantee that \( G^t \) stays bipartite, so Mini can’t utilize any strategy that requires her to create many odd cycles. Conversely, one can show that if Mini doesn’t try and create any odd cycles, then Max can play so that \( G^t \) ends with \( \frac{1}{4}n^2 \) edges. Thus any strategy of Mini’s giving a non-trivial bound has to attempt to make odd cycles, while making sure that the final graph ends up unbalanced if Max stops her from doing so.

Key idea: Mini will try and grow a bunch of long, edge-disjoint paths sharing a common endpoint.
The \( \{C_3, \ldots, C_{2k+1}\} \)-saturation game

The upper bound for \( \text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \) is significantly harder. We’ve shown that Max can guarantee that \( G^t \) stays bipartite, so Mini can’t utilize any strategy that requires her to create many odd cycles. Conversely, one can show that if Mini doesn’t try and create any odd cycles, then Max can play so that \( G^t \) ends with \( \frac{1}{4} n^2 \) edges. Thus any strategy of Mini’s giving a non-trivial bound has to attempt to make odd cycles, while making sure that the final graph ends up unbalanced if Max stops her from doing so.

Key idea: Mini will try and grow a bunch of long, edge-disjoint paths sharing a common endpoint. If she succeeds, she connects the paths together and forms many \( C_{2k+1} \)’s.
The \( \{C_3, \ldots, C_{2k+1}\}\)-saturation game

The upper bound for \( \text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \) is significantly harder. We’ve shown that Max can guarantee that \( G^t \) stays bipartite, so Mini can’t utilize any strategy that requires her to create many odd cycles. Conversely, one can show that if Mini doesn’t try and create any odd cycles, then Max can play so that \( G^t \) ends with \( \frac{1}{4}n^2 \) edges. Thus any strategy of Mini’s giving a non-trivial bound has to attempt to make odd cycles, while making sure that the final graph ends up unbalanced if Max stops her from doing so.

Key idea: Mini will try and grow a bunch of long, edge-disjoint paths sharing a common endpoint. If she succeeds, she connects the paths together and forms many \( C_{2k+1} \)’s. Conversely, if Max tries to destroy a path, the graph becomes more unbalanced.
The \( \{C_3, \ldots, C_{2k+1}\} \)-saturation game

Path Growing Phase 3:
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

Path Growing Phase 3:
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

Path Growing Phase 3:
The \( \{C_3, \ldots, C_{2k+1}\} \)-saturation game

Path Growing Phase 3:
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

Path Growing Phase 3:
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

Path Growing Phase 3:

Every time Max destroys paths, $|V^t|$ increases while $|U^t|$ stays the same.
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

Path Growing Phase 3:

Every time Max destroys paths, $|V^t|$ increases while $|U^t|$ stays the same. Thus eventually either $|V^t|$ becomes much larger than $|U^t|$ (in which case Mini maintains this), or Mini succeeds in making many long paths (which eventually she’ll connect to form $C_{2k+1}$’s).
The $\{C_3, \ldots, C_{2k+1}\}$-saturation game

For $k \geq 4$,

$$\left(\frac{1}{4} - \frac{1}{5k^2}\right) n^2 + o(n^2) \leq \text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \leq \left(\frac{1}{4} - \frac{1}{20^6 k^4}\right) n^2 + o(n^2).$$
The \( \{C_3, \ldots, C_{2k+1}\} \)-saturation game

For \( k \geq 4 \),
\[
\left( \frac{1}{4} - \frac{1}{5k^2} \right) n^2 + o(n^2) \leq \text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \leq \left( \frac{1}{4} - \frac{1}{20^6k^4} \right) n^2 + o(n^2).
\]

Conjecture

For all \( k \geq 1 \) there exists a \( c_k > 0 \) such that
\[
\text{sat}(\{C_3, \ldots, C_{2k+1}\}; n) \leq \left( \frac{1}{4} - c_k \right) n^2 + o(n^2).
\]
The \( \{C_3, \ldots, C_{2k+1}\} \)-saturation game

For \( k \geq 4 \),
\[
\left( \frac{1}{4} - \frac{1}{5k^2} \right) n^2 + o(n^2) \leq \text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n) \leq \left( \frac{1}{4} - \frac{1}{20^6 k^4} \right) n^2 + o(n^2).
\]

Conjecture

For all \( k \geq 1 \) there exists a \( c_k > 0 \) such that
\[
\text{sat}(\{C_3, \ldots, C_{2k+1}\}; n) \leq \left( \frac{1}{4} - c_k \right) n^2 + o(n^2).
\]

Conjecture

For all \( k \geq 2 \) and \( n \) sufficiently large,
\[
\text{sat}_g(\{C_3, \ldots, C_{2k-1}\}; n) \leq \text{sat}_g(\{C_3, \ldots, C_{2k+1}\}; n).
\]
The \((C_\infty \setminus \{C_3\})\)-saturation game

Let \(C_\infty = \{C_3, C_5, C_7, \ldots\}\). We wish to consider the \((C_\infty \setminus \{C_3\})\)-saturation game.
The \((C_\infty \setminus \{C_3\})\)-saturation game

Let \(C_\infty = \{C_3, C_5, C_7, \ldots\}\). We wish to consider the \((C_\infty \setminus \{C_3\})\)-saturation game.

**Theorem (S., 2019)**

\[
sat_g(C_\infty \setminus \{C_3\}; n) \leq 2n - 2.
\]
The \((C_\infty \setminus \{C_3\})\)-saturation game

Let \(C_\infty = \{C_3, C_5, C_7, \ldots\}\). We wish to consider the \((C_\infty \setminus \{C_3\})\)-saturation game.

**Theorem (S., 2019)**

\[
\text{sat}_g(C_\infty \setminus \{C_3\}; n) \leq 2n - 2.
\]

This in sharp contrast to the fact that \(\text{sat}_g(C_\infty; n) = \lfloor \frac{1}{4} n^2 \rfloor\).
Let $C_{\infty} = \{C_3, C_5, C_7, \ldots\}$. We wish to consider the $(C_{\infty} \setminus \{C_3\})$-saturation game.

**Theorem (S., 2019)**

\[
sat_g(C_{\infty} \setminus \{C_3\}; n) \leq 2n - 2.
\]

This in sharp contrast to the fact that $sat_g(C_{\infty}; n) = \left\lfloor \frac{1}{4} n^2 \right\rfloor$.

Key idea: Mini can play so that almost every edge of $G^t$ lies in a triangle.
The \((C_\infty \setminus \{C_3\})\)-saturation game

**Lemma**

If \(xy\) and \(xz\) are not in triangles, then \(yz\) is a legal move in the \((C_\infty \setminus \{C_3\})\)-saturation game.

![Graph Diagram]

By doing this repeatedly, Mini can guarantee that “most” edges are in triangles.
The \((\mathcal{C}_\infty \setminus \{C_3\})\)-saturation game

**Lemma**

If \(xy\) and \(xz\) are not in triangles, then \(yz\) is a legal move in the \((\mathcal{C}_\infty \setminus \{C_3\})\)-saturation game.

By doing this repeatedly, Mini can guarantee that “most” edges are in triangles.
Lemma

If $G$ is a graph where “most” edges are in triangles and $G$ contains no $C_k$ with $k \geq 5$ odd, then $G$ contains no $C_k$ with $k \geq 5$. 
The \((\mathcal{C}_\infty \setminus \{C_3\})\)-saturation game

**Lemma**

If \(G\) is a graph where “most” edges are in triangles and \(G\) contains no \(C_k\) with \(k \geq 5\) odd, then \(G\) contains no \(C_k\) with \(k \geq 5\).
The \((\mathcal{C}_{\infty} \setminus \{C_3\})\)-saturation game

**Lemma**

If \(G\) is a graph where “most” edges are in triangles and \(G\) contains no \(C_k\) with \(k \geq 5\) odd, then \(G\) contains no \(C_k\) with \(k \geq 5\).

**Lemma**

\[\text{ex}\left(\{C_5, C_6, C_7, \ldots\}, n\right) \leq 2n - 2.\]
The \((C_\infty \setminus \{C_3\})\)-saturation game

Theorem (S., 2019)

\[ \text{sat}_g(C_\infty \setminus \{C_3\}; n) \leq 2n - 2. \]
Theorem (S., 2019)

$$\text{sat}_g(C_\infty \setminus \{C_3\}; n) \leq 2n - 2.$$  

Proof.

Mini plays so that “most” edges are in triangles.
The \((C_\infty \setminus \{C_3\})\)-saturation game

Theorem (S., 2019)

\[
sat_g(C_\infty \setminus \{C_3\}; n) \leq 2n - 2.
\]

Proof.

Mini plays so that “most” edges are in triangles. This implies that the graph is \(C_k\)-free for all \(k \geq 5\), and thus has at most \(2n - 2\) edges at the end of the game.
The \((C_\infty \setminus \{C_3\})\)-saturation game

**Theorem (S., 2019)**

\[ \text{sat}_g(C_\infty \setminus \{C_3\}; n) \leq 2n - 2. \]

**Proof.**

Mini plays so that “most” edges are in triangles. This implies that the graph is \(C_k\)-free for all \(k \geq 5\), and thus has at most \(2n - 2\) edges at the end of the game.

**Theorem (English-Masarik-Meger-McCourt-Ross-S., 2019+)**

\[ \text{sat}_g(C_\infty \setminus \{C_3, \ldots, C_{2k+1}\}; n) = O_k(n). \]
The $(C_\infty \setminus \{C_3\})$-saturation game

**Theorem (S., 2019)**

\[ \text{sat}_g(C_\infty \setminus \{C_3\}; n) \leq 2n - 2. \]

**Proof.**

Mini plays so that “most” edges are in triangles. This implies that
the graph is $C_k$-free for all $k \geq 5$, and thus has at most $2n - 2$
edges at the end of the game.


\[ \text{sat}_g(C_\infty \setminus \{C_3, \ldots, C_{2k+1}\}; n) = O_k(n). \]

Key idea: Mini plays so that “most” edges are in odd cycles.
The \((C_\infty \setminus \{C_3\})\)-saturation game

We can prove linear bounds for “sparser” families of odd cycles.
We can prove linear bounds for “sparser” families of odd cycles.

**Theorem (EMMRS, 2019+)**

Let $C = \bigcup_{r \geq 0} \{ C_{3^r+4} \} = \{ C_5, C_7, C_{13}, C_{31}, C_{85} \ldots \}$. 
We can prove linear bounds for “sparser” families of odd cycles.

**Theorem (EMMRS, 2019+)**

Let $C = \bigcup_{r \geq 0} \{C_{3r+4}\} = \{C_5, C_7, C_{13}, C_{31}, C_{85} \ldots\}$. Then

$$\text{sat}_g(C; n) \leq 2n - 2$$
The \((C_{\infty} \setminus \{C_3\})\)-saturation game

We can prove linear bounds for “sparser” families of odd cycles.

Theorem (EMMRS, 2019+)

Let \(C = \bigcup_{r \geq 0} \{C_3r+4\} = \{C_5, C_7, C_{13}, C_{31}, C_{85} \ldots\}\). Then

\[
sat_g(C; n) \leq 2n - 2,
\]

but we also have

\[
sat_g(C \cup \{C_3\}; n) \geq \frac{6}{25} n^2 + o(n^2).
\]
We can prove linear bounds for “sparser” families of odd cycles.

**Theorem (EMMRS, 2019+)**

Let $\mathcal{C} = \bigcup_{r \geq 0} \{C_3^r + 4\} = \{C_5, C_7, C_{13}, C_{31}, C_{85}, \ldots\}$. Then

$$\text{sat}_g(\mathcal{C}; n) \leq 2n - 2,$$

but we also have

$$\text{sat}_g(\mathcal{C} \cup \{C_3\}; n) \geq \frac{6}{25} n^2 + o(n^2).$$

In particular, $\text{sat}_g(\mathcal{C}; n)$ is very sensitive to small cycles.
The \((C_\infty \setminus \{C_3\})\)-saturation game

**Conjecture (S., 2019)**

\[ \text{sat}_g(C_\infty \setminus \{C_3\}; n) \sim 2n. \]
The \((C∞ \setminus \{C_3\})\)-saturation game

**Conjecture (S., 2019)**

\[
sat_g(C∞ \setminus \{C_3\}; n) \sim 2n.
\]

**Theorem (EMMRS, 2019+)**

\[
sat_g(C∞ \setminus \{C_3\}; n) \geq \frac{55}{34} n - O(1)
\]
The \((\mathcal{C}_\infty \setminus \{C_3\})\)-saturation game

**Conjecture (S., 2019)**

\[ \text{sat}_g(\mathcal{C}_\infty \setminus \{C_3\}; n) \sim 2n. \]

**Theorem (EMMRS, 2019+)**

\[ \text{sat}_g(\mathcal{C}_\infty \setminus \{C_3\}; n) \geq \frac{55}{34}n - O(1) \geq 1.61n - O(1) \]

Neither of these bounds are tight.
The \((C_{\infty} \setminus \{C_3\})\)-saturation game

**Conjecture (S., 2019)**

\[
sat_g(C_{\infty} \setminus \{C_3\}; n) \sim 2n.
\]

**Theorem (EMMRS, 2019+)**

\[
sat_g(C_{\infty} \setminus \{C_3\}; n) \geq \frac{55}{34} n - O(1) \geq 1.61n - O(1),
\]

\[
sat_g(C_{\infty} \setminus \{C_3\}; n) \leq \left(2 - \frac{1}{196}\right)n + O(1)
\]

Neither of these bounds are tight.
The \((C_\infty \setminus \{ C_3 \})\)-saturation game

**Conjecture (S., 2019)**

\[
\text{sat}_g(C_\infty \setminus \{ C_3 \}; n) \sim 2n.
\]

**Theorem (EMMRS, 2019+)**

\[
\text{sat}_g(C_\infty \setminus \{ C_3 \}; n) \geq \frac{55}{34} n - O(1) \geq 1.61n - O(1),
\]

\[
\text{sat}_g(C_\infty \setminus \{ C_3 \}; n) \leq \left(2 - \frac{1}{196}\right) n + O(1) \leq 1.995n + O(1).
\]

Neither of these bounds are tight.
The \((C_\infty \setminus \{C_3\})\)-saturation game

**Conjecture (S., 2019)**

\[
\text{sat}_g(C_\infty \setminus \{C_3\}; n) \sim 2n.
\]

**Theorem (EMMRS, 2019+)**

\[
\begin{align*}
\text{sat}_g(C_\infty \setminus \{C_3\}; n) &\geq \frac{55}{34} n - O(1) \geq 1.61n - O(1), \\
\text{sat}_g(C_\infty \setminus \{C_3\}; n) &\leq \left(2 - \frac{1}{196}\right) n + O(1) \leq 1.995n + O(1).
\end{align*}
\]

**Conjecture (S., 2019+)**

*Neither of these bounds are tight.*
We’ve shown that \( \text{sat}_g(\mathcal{C}_\infty \setminus \{ C_3 \}; n) = O(n) \) and that
\( \text{sat}_g(\mathcal{C}_\infty \setminus \{ C_{2k+1} \}; n) = \Omega(n^2) \) for all \( k \geq 3 \).
The \((C_\infty \setminus \{C_3\})\)-saturation game

We’ve shown that \(\text{sat}_g(C_\infty \setminus \{C_3\}; n) = O(n)\) and that \(\text{sat}_g(C_\infty \setminus \{C_{2k+1}\}; n) = \Omega(n^2)\) for all \(k \geq 3\).

**Question**

What is the order of magnitude of \(\text{sat}_g(C_\infty \setminus \{C_5\}; n)\)?
The $(C_\infty \setminus \{C_3\})$-saturation game

We’ve shown that $\text{sat}_g(C_\infty \setminus \{C_3\}; n) = O(n)$ and that $\text{sat}_g(C_\infty \setminus \{C_{2k+1}\}; n) = \Omega(n^2)$ for all $k \geq 3$.

**Question**

What is the order of magnitude of $\text{sat}_g(C_\infty \setminus \{C_5\}; n)$?

**Question**

What other families of cycles $C$ have (non-)linear game saturation number?
The End

Thank You!