Odd Cycle Saturation Games

Sam Spiro, UC San Diego.

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The $\mathcal{F}$-saturation game

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The game starts with an initially empty graph $G$ on $n$ vertices. Max and Mini alternate turns adding a new edge to $G$, with the only restriction being that neither play can add an edge that would create some $F \in \mathcal{F}$ as a subgraph in $G$. When the game ends, Max gets a point for every edge in $G$ at the end of the game and Mini loses a point for every edge in $G$. Thus Max wants the game to last as long as possible, while Mini wants the game to end as quickly as possible.
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The $\mathcal{F}$-saturation number

Let $\text{sat}_g(\mathcal{F}; n)$ denote the number of edges in $G$ at the end of the $\mathcal{F}$-saturation game when both players play optimally. The goal is to find this value, which is known as the $\mathcal{F}$-saturation number.
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Remark

Technically the $\mathcal{F}$-saturation game, and hence $\text{sat}_g(\mathcal{F}; n)$, depends on which player starts the game. This choice won’t effect any of our results, but for concreteness we’ll assume Max adds the first edge.
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Example

$$n - 1 \leq \text{sat}_g(\{C_3\}; n) \leq \frac{1}{4} n^2.$$
The $\mathcal{F}$-saturation number

**Theorem (Furedi, Reimer, Sersess, 1992)**

\[
\text{sat}_g(\{C_3\}; n) \geq \frac{1}{2} n \log n + o(n \log n).
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**Theorem (Biró, Horn, Wildstrom, 2014)**

$$\text{sat}_g(\{C_3\}; n) \leq \frac{26}{121} n^2 + o(n^2).$$

These are the only known bounds for the triangle-free game. Our goal is to establish a lower bound for a related game, namely the $\{C_3, C_5\}$-saturation game.

Key idea: Max can force the graph to be bipartite throughout this game.
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The \( \{ C_3, C_5 \} \)-saturation game

In general, let \( X^t \) denote \( X \) after \( t \) edges have been added in the game, e.g. \( G^t \) denotes the graph after \( t \) edges have been played, \( e^t \) denotes the edge added at time \( t \), etc.
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In general, let $X^t$ denote $X$ after $t$ edges have been added in the game, e.g. $G^t$ denotes the graph after $t$ edges have been played, $e^t$ denotes the edge added at time $t$, etc. Max wishes to end each of his turns such that $G^t$ satisfies the following conditions.

(1*) Let $e_1 = uv$ denote the first edge played in the game. $G^t$ contains exactly one non-trivial connected component, and this component is bipartite with parts $U^t \ni u$ and $V^t \ni v$.

(2*) Every vertex of $U^t \cup V^t$ is adjacent to a vertex in $U^t_0 \cup V^t_0$. How can Max play so that he can achieve this?
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Let \( U^t_0 = N^t(v) \) (the good vertices), and \( U^t_1 = U^t \setminus U^t_0 \) (the bad vertices). Define an analogous partition for \( V^t \).
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Inductively assume that Max plays so \( G^{t-2} \) satisfies (1*) and (2*).
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Inductively assume that Max plays so \( G^{t-2} \) satisfies (1*) and (2*). What if \( e^{t-1} = xv', \ x \notin U^{t-2} \cup V^{t-2}, \ v' \in V^{t-2} \) (Add to \( U \))?
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Inductively assume that Max plays so \( G^{t-2} \) satisfies (1*) and (2*). What if \( e^{t-1} = u'u'' \), \( u', u'' \in U^{t-2} \)?
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Lemma

Let $t$ be such that $G^t$ satisfies (1*) and (2*). Then $U^{t+1}$ and $V^{t+1}$ are independent sets for any valid choice of $e^{t+1}$ in the $\{C_3, C_5\}$-saturation game for $k \geq 2$. 

**Proof.**

$U^t$ and $V^t$ are independent sets since $G^t$ satisfies (1*). Assume $e^{t+1} = v'v''$ with $v', v'' \in V^t$. Thus having $e^{t+1} = v'v''$ would create either a $C_3$ or a $C_5$ since $d^t(v', v'')$ is even, which is forbidden.

Given this lemma, Mini can only do Internal, Outside, and Add to $U/V$ moves, so Max can indeed play so that (1*) and (2*) are maintained.
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\[
d^t(v', v'') \leq d^t(v', u') + d^t(u', v) + d^t(v, u'') + d^t(u'', v'') = 4.
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\[(3^*) \quad b_U^t := |V_1^t| + (|U^t| - 2|V^t|) \leq 0,
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\(b_V^t := |U_1^t| + (|V^t| - 2|U^t|) \leq 0.\)
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The idea with this property is that \(|U^t|\) and \(|V^t|\) are always within a factor of two of each other.
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The idea with this property is that $|U^t|$ and $|V^t|$ are always within a factor of two of each other. Further, if $|U^t|$ is much larger than $|V^t|$, then there must be few bad $V_1^t$ vertices.
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If Mini does an Internal or Outside move then Max acts as he did before, and with this \(b_U^t, b_V^t\) don’t increase.
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If Mini does an Internal or Outside move then Max acts as he did before, and with this $b_U^t, b_V^t$ don’t increase. However, Max has to be more careful when Mini plays an Add to $U$ move.
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Case 1: \(|U^{t+1}| \leq 2|V^{t+1}|\).

\[
\begin{align*}
& b_U^t = |V_1^t| + (|U^t| - 2|V^t|) = 0, \\
& b_V^t = |U_1^t| + (|V^t| - 2|U^t|) = -5.
\end{align*}
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Case 1: \(|U^{t+1}| \leq 2|V^{t+1}|\).

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b_U^{t+2} = |V_1^{t+2}| + (|U^{t+2}| - 2|V^{t+2}|) = 0, \\
b_V^{t+2} = |U_1^{t+2}| + (|V^{t+2}| - 2|U^{t+2}|) = -6.
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Case 2: $|U^{t+1}| > 2|V^{t+1}|$.

$$b_U^t = |V_1^t| + (|U^t| - 2|V^t|) = 0,$$
$$b_V^t = |U_1^t| + (|V^t| - 2|U^t|) = -7.$$
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Theorem (S., 2018)

$$\text{sat}_g(\{C_3, C_5\}; n) \geq \frac{2}{9} n^2 + o(n^2).$$
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**Theorem (S., 2018)**

$$\text{sat}_g(\{C_3, C_5\}; n) \geq \frac{2}{9} n^2 + o(n^2).$$

**Proof.**

Max follows the strategy defined beforehand as long as there exists isolated vertices in $G^t$, afterwards he plays arbitrarily.
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**Proof.**

Max follows the strategy defined beforehand as long as there exists isolated vertices in $G^t$, afterwards he plays arbitrarily. At the end of the game, $G$ will be a complete bipartite graph with, say, $|V| \leq |U| \leq 2|V| + 1$, and hence contains at least $\frac{2}{9} n^2 + o(n^2)$ edges.
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**Theorem (S., 2018)**

$$\text{sat}_g(\{C_3, C_5\}; n) \geq \frac{6}{25} n^2 + o(n^2).$$
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Essentially one uses the same strategy as before but with a stronger induction. Namely, Max maintains the following.
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Essentially one uses the same strategy as before but with a stronger induction. Namely, Max maintains the following.

$$b_U^t := |V_1^t| + (|U^t| - \frac{3}{2}|V^t| - 2) \leq 0,$$
$$b_V^t := |U_1^t| + (|V^t| - \frac{3}{2}|U^t| - 2) \leq 0.$$

$$b_U^t + b_V^t \leq -2.$$
Improving the constant

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\[(4^*)\]

\[b_U^t + b_V^t \leq -2.\]

The main idea is that $(4^*)$ guarantees that one of $b_U^t, b_V^t \leq -1$, and hence one of the sets $U^t, V^t$ can afford to have its structure disrupted.
The $C_{2k+1}$-saturation game

Can Max do better if we forbid larger cycles?
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Can Max do better if we forbid larger cycles? Let $\mathcal{C}_{2k+1} = \{C_3, C_5, \ldots, C_{2k+1}\}$. We claim that Max can use the same strategy as before to get $\text{sat}_g(\mathcal{C}_{2k+1}; n) \geq \frac{6}{25} n^2 + o(n^2)$ for all $k \geq 2$. Can we do better with our additional structure?
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Theorem (S., 2018)

For $k \geq 4$,

$$\left(\frac{1}{4} - \frac{1}{5k^2}\right) n^2 + o(n^2) \leq \text{sat}_g(C_{2k+1}; n)$$
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$$\left( \frac{1}{4} - \frac{1}{5k^2} \right) n^2 + o(n^2) \leq \text{sat}_g(C_{2k+1}; n) \leq \left( \frac{1}{4} - \frac{1}{20^6 k^4} \right) n^2 + o(n^2).$$
The $C_{2k+1}$-saturation game

Can Max do better if we forbid larger cycles? Let $C_{2k+1} = \{C_3, C_5, \ldots, C_{2k+1}\}$. We claim that Max can use the same strategy as before to get $\text{sat}_g(C_{2k+1}; n) \geq \frac{6}{25} n^2 + o(n^2)$ for all $k \geq 2$. Can we do better with our additional structure?

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Idea for the lower bound: call a vertex bad if it’s roughly distance $k$ away from $u$ or $v$ (as opposed to those that simply aren’t adjacent to $u/v$).
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Idea for the lower bound: call a vertex bad if it’s roughly distance $k$ away from $u$ or $v$ (as opposed to those that simply aren’t adjacent to $u/v$). By being more careful in the previous argument, and by making a slight tweak to the strategy, one can replace the $\frac{3}{2}$ we had before with $\gamma_k \to 1$. 

The $C_{2k+1}$-saturation game

The upper bound for $\text{sat}_g(C_{2k+1}; n)$ is significantly harder.
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The upper bound for $\text{sat}_g(C_{2k+1}; n)$ is significantly harder. We’ve shown that Max can guarantee that $G^t$ stays bipartite, so Mini can’t utilize any strategy that requires her to create many odd cycles.

Conversely, one can show that if Mini doesn’t try and create any odd cycles, then Max can play so that $G^t$ ends with $\frac{14}{2}n^2$ edges. Thus any strategy of Mini’s giving a non-trivial bound has to attempt to make odd cycles, while making sure that the final graph ends up unbalanced if Max stops her from doing so.

Key idea: Mini will try and grow a bunch of long, edge-disjoint paths sharing a common endpoint. If she succeeds, she connects the paths together and forms many $C_{2k+1}$’s. Conversely, if Max tries to destroy a path, the graph becomes more unbalanced.
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The $C_{2k+1}$-saturation game

Path Growing Phase 3:
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The $C_{2k+1}$-saturation game

Path Growing Phase 3:

Every time Max destroys paths, $|V^t|$ increases while $|U^t|$ stays the same. Thus eventually either $|V^t|$ becomes much larger than $|U^t|$ (in which case Mini maintains this), or Mini succeeds in making many long paths (which eventually she’ll connect to form $C_{2k+1}$’s).
The $C_{2k+1}$-saturation game

For $k \geq 4$,

$$\left(\frac{1}{4} - \frac{1}{5k^2}\right) n^2 + o(n^2) \leq \text{sat}_g(C_{2k+1}; n) \leq \left(\frac{1}{4} - \frac{1}{20^6 k^4}\right) n^2 + o(n^2).$$
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What is the constant in the $C_{2k+1}$-saturation game?
The $C_{2k+1}$-saturation game

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What is the constant in the $C_{2k+1}$-saturation game?

**Conjecture**

For all $k \geq 1$ there exists a $c_k > 0$ such that

\[
\text{sat}(C_{2k+1}; n) \leq \left( \frac{1}{4} - c_k \right) n^2 + o(n^2).
\]
The $C_{2k+1}$-saturation game

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Conjecture

For all $k \geq 2$ and $n$ sufficiently large,
\[
\text{sat}_g(C_{2k-1}; n) \leq \text{sat}_g(C_{2k+1}; n).
\]
The \((C_\infty \setminus \{C_3\})\)-saturation game

Let \(C_\infty = \{C_3, C_5, C_7, \ldots\}\). We wish to consider the \((C_\infty \setminus \{C_3\})\)-saturation game.
The \((C_\infty \setminus \{C_3\})\)-saturation game

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**Theorem (S., 2018)**

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\text{sat}_g(C_\infty \setminus \{C_3\}; n) \leq 2n - 2.
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The (\(C_\infty \setminus \{C_3\}\))-saturation game
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This in sharp contrast to the fact that $\text{sat}_g(C_\infty; n) = \frac{1}{4}n^2$.

Key idea: Mini can play so that almost every edge of $G^t$ lies in a triangle.
The \((C_\infty \setminus \{C_3\})\)-saturation game

We will say that a vertex \(v\) is good if all but at most one edge incident to \(v\) is contained in a triangle.
The \((C_\infty \setminus \{C_3\})\)-saturation game

We will say that a vertex \(v\) is good if all but at most one edge incident to \(v\) is contained in a triangle. We will say that a graph \(G\) is \(k\)-good if there exists a set of edges \(B(G)\) with \(|B(G)| \leq k\) such that every vertex of \(G - B(G)\) is good.
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**Proposition**

*Mini can play in the \((C_\infty \setminus \{C_3\})\)-saturation game so that she ends each of her turns with \(G^t\) being 1-good.*
The \((C_\infty \setminus \{C_3\})\)-saturation game

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**Proposition**

*Mini can play in the \((C_\infty \setminus \{C_3\})\)-saturation game so that she ends each of her turns with \(G^t\) being 1-good.*

**Lemma**

*If \(G\) is a 2-good graph that contains no \(C_{2k+1}\) for any \(k \geq 2\), then \(G\) contains no \(C_\ell\) for any \(\ell \geq 5\).*
Inductively assume that Mini has played so that $G^{t-2}$ is 1-good.
The \((C_\infty \setminus \{C_3\})\)-saturation game

Inductively assume that Mini has played so that \(G^{t-2}\) is 1-good. This means that \(G^{t-1}\) is 2-good (and hence contains no large cycles).
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The \((C_{\infty} \setminus \{C_3\})\)-saturation game

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Claim

\(e^t = uw\) is a legal move.
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**Claim**

$e^t = uw$ is a legal move.

With this $vu$ and $vw$ are both contained in triangles, and one can show that this implies that $G^t$ is 1-good.
We’ve now shown that Mini can maintain that $G^t$ is 1-good whenever she ends her turn, and hence $G^t$ is 2-good for all $t$. 

Lemma \[ \text{ex}(n, \{C_5, C_6, \ldots\}) \leq 2n - 2, \text{ and this is sharp when } n \equiv 1 \text{ mod } 3. \]

With this we have that $\text{sat}_g(C_\infty \setminus \{C_3\}; n) \leq 2n - 2.$
The \((C_\infty \setminus \{C_3\})\)-saturation game

We’ve now shown that Mini can maintain that \(G^t\) is 1-good whenever she ends her turn, and hence \(G^t\) is 2-good for all \(t\). Recall that this implies that \(G^t\) contains no \(C_\ell\) with \(\ell \geq 5\).
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The \((C_\infty \setminus \{C_3\})\)-saturation game

**Conjecture**

\[ \text{sat}_g(C_\infty \setminus \{C_3\}; n) \sim 2n. \]
The \((C_\infty \setminus \{C_3\})\)-saturation game

**Conjecture**

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sat_g(C_\infty \setminus \{C_3\}; n) \sim 2n.
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Implicitly we’ve shown that \(sat_g(C_\infty \setminus \{C_{2k+1}\}; n) = \Omega(n^2)\) for all \(k \geq 3\), and we’ve just seen that \(sat_g(C_\infty \setminus \{C_3\}; n)\) is linear.
### The $(C_\infty \setminus \{C_3\})$-saturation game

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**Question**

What is the order of magnitude of $\text{sat}_g(C_\infty \setminus \{C_5\}; n)$?
The \( (C_\infty \setminus \{C_3\}) \)-saturation game

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Thank You!