Slow Fibonacci Walks

Sam Spiro, UC San Diego.

INSERT DATE
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It gives you the chance to practice giving talks, especially ones geared towards a general mathematical audience (e.g. job talks).

It gives you an excuse to (better) learn a topic (e.g. for a qual class, or on a research topic you might want to explore).

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Given positive integers $a_1, a_2$, we define the $(a_1, a_2)$-Fibonacci walk to be the sequence $w = w(a_1, a_2)$ satisfying $w_1 = a_1, w_2 = a_2, w_k+2 = w_k+1 + w_k$.

For example, if $w = w(10, 2)$, this gives the sequence $10, 2, 12, 14, 26, 40, 66, ...$

We say that $w$ is an $n$-Fibonacci walk if $w_s = n$ for some $s$. For example, the above $w$ is a 40-Fibonacci walk.
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While that does look pretty interesting, we’ll instead make our walk “slower.” To this end, we will say that a sequence $w_k$ is an $n$-slow Fibonacci walk if $w_s = n$ and $s$ is as large as possible. For example, the following are all 40-Fibonacci walks.

- $1024, 40, 1064 \ldots$
- $8, 8, 16, 24, 40, 64 \ldots$
- $5, 5, 10, 15, 25, 40, 65 \ldots$
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While that does look pretty interesting, we’ll instead make our walk “slower.” To this end, we will say that a sequence \( w_k \) is an \( n \)-slow Fibonacci walk if \( w_s = n \) and \( s \) is as large as possible. For example, the following are all 40-Fibonacci walks.

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\end{align*}
\]

However, the first two can’t be slow (since the next two achieve 40 with \( s = 6 \)), and one can verify that \( w_k(5, 5) \) and \( w_k(10, 2) \) are (the unique) 40-slow Fibonacci walks.
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**Lemma**

\[ s(n) = 2 \text{ iff } n = 1, \text{ in which case } (x, a) \text{ is a } 1\text{-good pair for all } x. \]
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Proof.

Otherwise \(b - a \geq 1\) and \(w_{s+1}(a - b, b) = w_s(b, a) = n\), a contradiction to the definition of \(s\).
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Lemma

\[ w_k(b, a) = af_{k-1} + bf_{k-2}, \text{ where } f_k \text{ denotes the } k\text{th Fibonacci number.} \]
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Lemma

Let \( s = s(n) > 2 \). If \((b, a)\) is \(n\)-good, then \((b', a')\) is \(n\)-good iff \( a' = a + kf_{s-2} \geq 1 \) and \( b' = b - kf_{s-1} \geq 1 \) for some \( k \).
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Proof.

By the above lemma, every \( n \)-good pair is a solution to the diophantine equation \( xf_{s-1} + yf_{s-2} = n \), and the result follows since \( \gcd(f_{s-1}, f_{s-2}) = 1 \) for \( s > 2 \). \( \square \)
# Slow Fibonacci Walks

<table>
<thead>
<tr>
<th>Theorem (Jones, Kiss (1998); Chung, Graham, S. (2019))</th>
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\[
0 < (\phi)^{-s+1}(\phi b - a) < 1,
\]

which is true when \( s \) is odd for \( 1 \leq a \leq b \leq f_t \).
Let $p(n)$ denote the number of $n$-good pairs. Corollary $p(n) \leq 2$, with equality iff $a(n) > fs - 2$.

Proof. Let $1 \leq a \leq b \leq fs - 1$ be as in the previous theorem. Recall that every $n$-good pair is of the form $(b', a') = (b + kf, a - kf)$ for some $k$ such that $b', a' \geq 1$.

Because $b \leq fs - 1$, we need $k \geq 0$ to have $b' \geq 1$. Because $a \leq fs - 1 \leq 2fs - 2$, we need $k \leq 1$.

Thus only the pairs with $k = 0, 1$ can work, and these both work iff $a > fs - 2$. 
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Given $n$, let $c, p$ be such that $n = \frac{1}{\sqrt{5}} c \phi^p$ with $\frac{1}{\sqrt{5}} \leq c < \frac{1}{\sqrt{5}} \phi$. Then

$T(n) = \begin{cases} 
\frac{1}{2\sqrt{5} \phi^4 c} + O(n^{-1/2}) & p \equiv 1 \mod 2, \\
\frac{\sqrt{5}}{2} c + \frac{1+\phi^{-5}}{2\sqrt{5}c} - 1 + O(n^{-1/2}) & p \equiv 0 \mod 2, c \leq \frac{1+\phi^{-3}}{\sqrt{5}}, \\
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**Proof**

Don't count \( m \leq n \) which have two pairs, instead count triples \((s(m), a(m), b(m)) = (s, a, b)\). These satisfy \( af_s - 1 + bf_s - 2 \leq n\), \( 1 \leq a \leq b \leq f_s - 1\), and \( a > f_s - 2\).

If \( s \) is such that \( n < f_s - 2 (f_s - 1 + f_s - 2) \approx \frac{1}{5} \phi^{2s - 2} \) then no such \( a, b \) exist.

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If \( n \) is in between these two values then things are annoying but doable.
“Applications”

\[ T(n) = \begin{cases} 
\frac{1}{2\sqrt{5}\phi^4c} + O(n^{-1/2}) & p \equiv 1 \mod 2, \\
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Theorem (Chung, Graham, S. (2019))

Let $D(n) = n - 1 \mid D \cap [n]$. Then $D(n) =$ \begin{align*}
\begin{cases}
\frac{1}{5} \phi q + 1 + \phi q + 1 / 10 \sqrt{5} n + O\left(\frac{n}{2}\right) & \text{if } q \equiv 1 \mod 4, \\
\frac{1}{5} \phi q + 2 & \text{if } q \equiv 3 \mod 4,
\end{cases}
\end{align*}

where $1 - \sqrt{5} n^2 \phi q + 1 - \phi q + 1 / 10 \sqrt{5} n + O\left(\frac{n}{2}\right)$.

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**Theorem (Chung, Graham, S. (2019))**

Let $D(n) = n^{-1}|D \cap [n]|$. Then

$$D(n) = \begin{cases} \frac{\sqrt{5}n}{2\phi + 1} + \frac{\phi^{q+1}}{10\sqrt{5}n} + O(n^{-1/2}) & \frac{1}{5}\phi^q \leq n < \frac{1}{5}\phi^{q+2}, q \equiv 1 \mod 4, \\ 1 - \frac{\sqrt{5}n}{2\phi + 1} - \frac{\phi^{q+1}}{10\sqrt{5}n} + O(n^{-1/2}) & \frac{1}{5}\phi^q \leq n < \frac{1}{5}\phi^{q+2}, q \equiv 3 \mod 4. \end{cases}$$
“Applications”

(a) Data plot of $D(n)$.

(b) Theory plot of $D(n)$. 
This says that roughly half the integers have \( w_{s+1} \) rounding down and half have them rounding up.
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\{d_{k+2} - d_k : k \geq 1\} = \{2, 3, 4, 5, 6, 8, 10\}.
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We know that \( w_{s+1} = \lfloor \phi n \rfloor \) or \( w_{s+1} = \lceil \phi n \rceil \). Intuitively, the smaller \( \phi n - \lfloor \phi n \rfloor \) is, the more likely it is that \( w_{s+1} = \lfloor \phi n \rfloor \).
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**Theorem (Chung, Graham, S. (2019))**

For \( d \leq \frac{1}{2} \), let \( P(n, d) = n^{-1}|\{m \leq n : m \text{ is } d-\text{paradoxical}\}| \).

Given \( n \), let \( c, p \) be such that \( n = \frac{1}{\sqrt{5}} c \phi^p \) with \( \frac{1}{\sqrt{5}} \leq c < \frac{1}{\sqrt{5}} \phi \). We have \( P(n, d) = 0 \) if \( d \leq \frac{1}{\sqrt{5}} \phi^{-1} \), and otherwise \( P(n, d) \) satisfies

\[
\begin{align*}
-\frac{1}{2} \phi^{-1} c + d + \left( d^2 - d + \frac{1}{2\sqrt{5}} \phi^{-1} \right) c^{-1} + O(n^{-1/2}) & \quad p \text{ odd, } c \leq \phi d, \\
\frac{\sqrt{5}}{2} \phi \left( d - \frac{1}{\sqrt{5}} \phi^{-1} \right)^2 c^{-1} + O(n^{-1/2}) & \quad p \text{ odd, } c \geq \phi d, \\
-\frac{1}{2} c + d + \left( \phi^{-1} d^2 - \phi^{-1} d + \frac{1}{2\sqrt{5}} \phi^{-2} \right) c^{-1} + O(n^{-1/2}) & \quad p \text{ even, } c \leq d, \\
\frac{\sqrt{5}}{2} \left( d - \frac{1}{\sqrt{5}} \phi^{-1} \right)^2 c^{-1} + O(n^{-1/2}) & \quad p \text{ even, } d \leq c \leq 1 - d \\
\frac{1}{2} c + d - 1 + \left( \phi d^2 - \phi d + \frac{1}{2\sqrt{5}} \phi^2 \right) c^{-1} + O(n^{-1/2}) & \quad p \text{ even, } c \geq 1 - d.
\end{align*}
\]
“Applications”

(a) Data plot of $P(n, .5)$.
(b) Theory plot of $P(n, .5)$.
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Note that when $\alpha = \beta = 1$ we have $g_k = f_k$, $\gamma = \phi$, $\lambda = -\phi^{-1}$. 
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**Theorem (S. (2019+))**

If \( s(n) > 2 \) and \( \beta = 1 \), there exist unique integers \( a = a(n), \ b = b(n) \) such that \( n = ag_{s-1} + bg_{s-2} \) and \( 1 \leq a \leq \alpha b \leq \alpha g_{s-1} \). Moreover, \((b, a)\) is \( n \)-good and \( w_{s+1}(b, a) \) is either \( \lfloor \gamma n \rfloor \) or \( \lceil \gamma n \rceil \).
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When \( \beta = 1 \) things get a bit more complicated.

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Generalized Walks: \( w_{k+2} = \alpha w_{k+1} + \beta w_k \)

As before we can use this theorem to prove a number of results about slow walks.
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**Theorem (S. (2019+))**

- If \( s(n) > 2 \), then
  \[ p(n) \leq \alpha^2 + 2\beta - 1. \]

  Moreover, there always exists an \( n \) achieving this.
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- *There exist infinitely many* \( n \) *with*
  \[
  p(n) = \lceil \gamma^2 \rceil - 1 = \alpha^2 + \beta + \lceil \alpha \beta \gamma^{-1} \rceil - 1,
  \]

  *and only finitely many* \( n \) *with*
  \[
  p(n) \geq \lceil \gamma^2 \rceil = \alpha^2 + \beta + \lceil \alpha \beta \gamma^{-1} \rceil.
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Let $S_p$ denote the set of $n$ with $p(n) > p$. A natural question to ask at this point is “given how ugly the Fibonacci density results were, surely Sam didn’t try and prove a generalized version of them?” Unfortunately, this conjecture is false.

**Theorem (S. (2019+))**

Given an integer $p$, let $d$ denote the smallest integer such that

$$\delta := \beta \gamma^{-1} p - \gamma d \leq \alpha.$$ 

If $\beta \leq p \leq \lceil \gamma^2 \rceil - 2$ and

$1 \leq c \leq (p - \beta + 1)\gamma / \alpha$,

then $n_{c,r}^{-1} | S_p \cap [n_{c,r}] | =$

$$c^{-1} \left( \frac{(2\beta - 2d - 1)\gamma (\alpha - 2\delta + \alpha^{-1}\delta^2)}{2\beta^2(\gamma^2 - 1)} + \frac{\gamma^2}{\gamma^2 - 1} \sum_{q=d+1}^{\beta-1} \frac{\beta - q}{\beta^2} \right) + O(\gamma^{-r} + (\beta \gamma^{-2})^r).$$
Generalized Walks: \( w_{k+2} = \alpha w_{k+1} + \beta w_k \)
Slowest Slow Walks

How slow is the slowest slow walk?

Define $s(n) = \max(\alpha, \beta) s_{\alpha,\beta}(n)$, as well as the pairs achieving this $S(n) = \{ (\alpha, \beta) : s_{\alpha,\beta}(n) = s(n) \}$.

A priori, any pair $(\alpha, \beta)$ could be an element of $S(n)$ for some $n$. However, it turns out that only a finite number of pairs have this property.

Theorem (S. (2019+))

Let $R = \{ (1,1), (2,1), (1,2), (1,3), (1,4) \}$.

For all $n > 1$, we have $S(n) \subseteq R$.

For all $(\alpha, \beta) \in R$, there exists an $n$ with $(\alpha, \beta) \in S(n)$ (possibly as large as $n = 10^{15}$ for the first appearance!).

The set of $n$ with $S(n) = \{ (1,1) \}$ has density 1.
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Let \( R = \{(1, 1), (2, 1), (1, 2), (1, 3), (1, 4)\} \).

- For all \( n > 1 \), we have \( S(n) \subseteq R \).
- For all \((\alpha, \beta) \in R\), there exists an \( n \) with \((\alpha, \beta) \in S(n)\) (possibly as large as \( n = 10^{15} \) for the first appearance!).
- The set of \( n \) with \( S(n) = \{(1, 1)\} \) has density 1.
More generally, given a set of pairs $T$, define $s_T(n) = \max_{(\alpha, \beta) \in T} s^{\alpha, \beta}(n)$, $S_T(n) = \{(\alpha, \beta) : s^{\alpha, \beta}(n) = s_T(n)\}$.
More generally, given a set of pairs $T$, define

$$s_T(n) = \max_{(\alpha, \beta) \in T} s^{\alpha,\beta}(n), \quad S_T(n) = \{(\alpha, \beta) : s^{\alpha,\beta}(n) = s_T(n)\}.$$ 

**Theorem**

There exists a finite set $R_T$ and number $n_T$ such that $S_T(n) \subseteq R_T$ for all $n \geq n_T$. Under certain conditions, $S_T(n)$ is almost always one specific pair.
Can you say anything more about how often each set appears in $S_T(n)$?
Can you say anything with $\alpha, \beta$ negative?
What happens with slow Tribonacci walks, i.e. $w_{k+3} = w_{k+2} + w_{k+1} + w_k$.

This was the greatest talk I've ever seen and the snacks were delightful. When can I give my own FFT talk and dazzle my friends and colleagues?

Proposition (Nikitopoulos, S. (2019)) Whenever you'd like!! +O(it's Friday at 12 and no one else has signed up yet).
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The End

Thank You!