Math 10A.
Midterm Exam 2
February 24, 2016

Turn off and put away your cell phone.

No calculators or other electronic devices are allowed during this exam.

You may use one page of notes, but no books or other assistance during this exam. If you violate these instructions or communicate in any way with any other student during this exam, you will receive a zero on the exam, and the zero will not be dropped when calculating your cumulative course average.

Read each question carefully, and answer each question completely.

Show all of your work; no credit will be given for unsupported answers.

Write your solutions clearly and legibly; no credit will be given for illegible solutions.

If any question is not clear, ask for clarification.

<table>
<thead>
<tr>
<th>#</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td></td>
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<tr>
<td>2</td>
<td>3</td>
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<tr>
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<td>30</td>
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</tbody>
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1. (12 points) Compute the derivative of the given function. You do not have to simplify your answers.

(a) \( y = \sqrt{x}(5^x + e^2) \)

**Hint:** apply product rule

\[
y'(x) = (\sqrt{x})' \cdot (5^x + e^2) + \sqrt{x} \cdot (5^x + e^2)'
\]

\[
(\sqrt{x})' = \frac{1}{2} \cdot x^{-\frac{1}{2}}
\]

\[
(5^x + e^2)' = (5^x)' + (e^2)' = (\ln 5) \cdot 5^x
\]

\[
y'(x) = \left(\frac{1}{2} \cdot x^{-\frac{1}{2}}\right)(5^x + e^2) + \sqrt{x} \cdot (\ln 5 \cdot 5^x)
\]

(b) \( y = \frac{1}{(x^2 + 3)^2} \)

**Hint:** apply chain rule

\[
x \rightarrow x^2 + 3 \rightarrow (x^2 + 3)^{-2} = f(g(x))
\]

\[
g(x) = x^2 + 3 \rightarrow x^2 \rightarrow (x^2 + 3)^{-2} = f(g(x))
\]

\[
\frac{d}{dx}(x^2 + 3) = 2x
\]

\[
y'(x) = (-2x)(x^2 + 3)^{-3}
\]

\[
y'(x) = 2x
\]

(c) \( y = 9^{x^6-4x} \)

**Hint:** apply chain rule

\[
x \rightarrow x^6 - 4x \rightarrow (x^6 - 4x)^{\frac{1}{2}} = f(g(x))
\]

\[
g(x) = x^6 - 4x \rightarrow x^6 \rightarrow 9^{x^6-4x} = f(g(x))
\]

\[
g'(x) = 6x^5 - 4 \quad \Rightarrow \quad f'(x) = (\ln 9) \cdot 9^{x^6-4x}
\]

\[
y'(x) = \left[9^{x^6-4x}\right]' = 9^{x^6-4x} \cdot (\ln 9) \cdot (6x^5 - 4)
\]

\[
= (\ln 9) \cdot 9^{x^6-4x} \cdot (6x^5 - 4)
\]
2. (3 points) Use the graph of $f$ below to determine which is larger in each of the following pairs:

(a) $f(1)$ or $f(3)$?

\[ f(1) < f(3) \quad f(x) \text{ is increasing} \]

(b) \[ \frac{f(2) - f(0)}{2 - 0} \quad \text{or} \quad \frac{f(4) - f(2)}{4 - 2} \]

\text{Hint: difference quotient represents the slope of segment connecting } (0, f(0)), (2, f(2)), \text{ and } (4, f(4)).

(c) $f'(1)$ or $f'(2)$?

\[ \frac{f(2) - f(0)}{2 - 0} > \frac{f(4) - f(2)}{4 - 2} \]

\text{Hint: derivative is slope of tangent line.}

\[ f(4) > f(2) \]
3. (8 points) Let \( f(x) = x^4 - 24x^2 \).

(a) Find the equation of the tangent line to the graph of \( f \) at the point at which \( x = 1 \).

\[
f(x) = 4x^3 - 48x
\]

at \( x = 1 \), \( f(1) = -23 \) \( f'(1) = -44 \)

**Hint:** tangent line at \( x = 1 \) passes through \((1, f(1))\), with slope \( f'(1) \)

\[
y = f'(1)(x-1) + f(1)
\]

\[
= (-44)(x-1) - 23
\]

(b) On which intervals is the graph of \( f \) is concave up?

**Hint:** Concave up: \( f''(x) > 0 \)

\[
f''(x) = 12x^2 - 48 > 0
\]

\[
12x^2 > 48
\]

\[
x^2 > 4
\]

\[
x > 2 \text{ or } x < -2
\]
4. (7 points) Below is the graph of the function $y = h(x)$. Use it to answer the following questions.

(a) List all values of $x$ at which the tangent line to the graph of $h$ is horizontal.

\[ \text{hint: tangent line is horizontal means } h'(x) = 0 \]
\[ x = -3, \]

(b) List all values of $x$ at which $h'(x)$ does not exist.

\[ x = -1, 3 \text{ on sharp corners, } h(x) \text{ does not exist} \]

(c) List all intervals on which $h'(x) < 0$.

\[ \text{hint: } h'(x) < 0 \text{ means } h(x) \text{ decrease} \]
\[ (-\infty, -3) \cup (-1, 3) \]

(d) List all intervals on which $h'$ is decreasing.

\[ \text{hint: } h' \text{ decrease means } h'' < 0 \text{ means grave down} \]
\[ (3, \infty) \]
1. (a) \( f(x) = x^2 \sin(2x) \)

hint: apply product rule
\[
\frac{d}{dx} (x^2 \sin(2x)) = (x^2)' \cdot \sin(2x) + x^2 \cdot \left( \frac{d}{dx} \sin(2x) \right)
\]
\[
= (2x) \cdot \sin(2x) + x^2 \cdot \left( 2 \cos(2x) \right)
\]

for \( \left( \sin(2x) \right)' \) apply chain rule
\[ g(x) = 2x \quad \Rightarrow \quad f(x) = \sin(2x) \quad \Rightarrow \quad f'(x) = \sin(2x)
\]

\[
\sin(2x)' = \left( f(g(x)) \right)' = f'(g(x)) \cdot g'(x) = 2 \cos(2x) \cdot 2 = 4 \cos(2x)
\]

\[
\therefore \quad \left( x^2 \sin(2x) \right)' = (2x) \sin(2x) + x^2 (2 \cos(2x))
\]

(b) \( \frac{\ln(2x)}{x} \)

hint: apply product rule
\[ \frac{\ln(2x)}{x} = \left( \frac{\ln(2x)}{x} \right) \cdot x^{-1}
\]

\[
\left( \left( \frac{\ln(2x)}{x} \right) \cdot x^{-1} \right)' = \left( \frac{\ln(2x)}{x} \right)' \cdot x^{-1} + \left( \frac{\ln(2x)}{x} \right) \cdot (x^{-1})'
\]

\[ \left( \frac{\ln(2x)}{x} \right)' = \left( \ln^2 + \ln x \right)' = (\ln^2)' + (\ln x)' = \frac{1}{x} - \frac{1}{x} \]

\[ \left( \frac{\ln(2x)}{x} \right)' = \frac{1}{x} \cdot \frac{1}{x} + \left( \frac{\ln(2x)}{x} \right) \cdot (1) \cdot x^{-2}
\]
2. \( F(2) = 3 \quad F(3) = 4 \quad G(7) = 2 \quad G'(7) = 8 \)
\( F(7) = 5 \quad F'(7) = 6 \)

(a) \( \mathcal{H}(x) = F(a(x)) \quad \mathcal{H}'(7) \)

apply chain rule: \( \mathcal{H}(x) = F'(a(x)) \cdot G'(x) \)

\[
\mathcal{H}'(7) = F'(G(7)) \cdot G'(7) = F'(2) \cdot 8 = 32
\]

(b) \( \mathcal{H}(7) \quad \mathcal{H}(x) = F(x) \cdot G(x) \)

\[
\mathcal{H}'(x) = F(x) \cdot G(x) + F(x) \cdot G'(x)
\]

\[
\mathcal{H}(7) = F(7) \cdot G(7) + F(7) \cdot G'(7) = 6 \cdot 2 + 5 \cdot 8 = 52
\]

(c) \( \mathcal{H}(7) \quad \mathcal{H}(x) = \frac{F(x)}{G(x)} \)

hint: apply quotient rule

\[
\mathcal{H}(x) = \frac{F(x) \cdot G(x) - F(x) \cdot G'(x)}{G^2(x)}
\]

\[
\mathcal{H}(7) = \frac{F(7) \cdot G(7) - F(7) \cdot G'(7)}{G^2(7)} = \frac{6 \cdot 2 - 5 \cdot 8}{8^2} = \frac{-28}{64}
\]
6. \( \frac{4}{2} \text{ hour per day} = \frac{2 \text{ days}}{2} \)

(6) on the 45th day, there is 11 more daylight.

(7) \( \frac{2 \text{ days}}{2} = \frac{2 \text{ days}}{2} \)

Thus, D(25) = 0.02 hours/day/day.

0.02 more hours/day/day. Tomorrow 0.02.

4. (a) on 25th day of the year, there will be

\[ f'(x) = \begin{cases} \frac{3}{x^2} + 6x & \text{for } x > 0 \\ 6x + 6 & \text{for } x < 0 \end{cases} \]

and \( f(x) \) decreasing.

(6) \( f'(x) < 0 \) decreasing.

\[ \begin{align*} &\text{for } x > 2, \quad 3x^2 + 6x = 3x(x + 2) > 0 \\ &\text{for } 0 < x < 2, \quad 3x^2 + 6x = 3x(x + 2) > 0 \\ &\text{for } x < 0, \quad 3x^2 + 6x = 3x(x + 2) < 0 \end{align*} \]

\[ f(x) = \begin{cases} \frac{x}{x + 3} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases} \]

and \( f(x) \) increasing.

(9) on what intervals is \( f(x) \) increasing.