Fall 2011 Final Exam Solutions

Dear Math 10A Students,

These are solutions to the Fall 2011 Math 10A final exam exam posted on the course website. Please only refer to these solutions after you’ve tried the problems on your own. If you have any questions regarding these questions or the solutions presented, please post them on Piazza. Good luck!

Question 1:
(a) \[ \lim_{c \to 0} \frac{x^2 + 3x}{2x^2 - 1} = \frac{0}{-1} = 0 \]
(b) \[ \lim_{x \to \infty} \sin(x) \] does not exist since \( \sin(x) \) oscillates between 1 and \(-1\) as \( x \to \infty \).
(c) \[ \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \left( \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} \right) = \lim_{x \to 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)} = \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{6} \]

Question 2:
(a) By the chain rule, \( p'(x) = f'(x) f(x) \). So, \( p'(3) = f'(3) f(3) = \frac{-5}{2} \).
(b) By the quotient rule, \( q'(x) = \frac{x f'(x) - f(x)}{x^2} \). So, \( q'(3) = \frac{3 f'(3) - f(3)}{9} = \frac{3(-5) - 2}{9} = -\frac{17}{9} \).

Question 3:
(a) \( f'(1) > f'(2) \).
(b) \( \frac{f(1) - f(0)}{1 - 0} > \frac{f(2) - f(1)}{2 - 1} \).
(c) \( \frac{f(2) - f(1)}{2 - 1} > f'(2) \).

Question 4:
We are asked to find constants \( a, b \) such that the given function is differentiable at the given points. Thus, we begin by recalling that a function \( f(x) \) is differentiable at \( x \) if
\[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
exists. Furthermore, a limit exists if and only if the left and right-hand limits are equal. So, the above limit exists precisely when \( \lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0^-} \frac{f(x + h) - f(x)}{h} \).

(a) As noted above, the function is differentiable at \( x = 0 \) if the following two limits are equal:
\[ \lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2 + 4h - 0}{h} = \lim_{h \to 0^+} h + 4 = 4 \]
\[ \lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^-} \frac{ah - 0}{h} = \lim_{h \to 0^-} a = a. \]
For these limits to be equal, we need \( a = 4 \).
(b) Similar to our approach in part (a), we find
\[
\lim_{h \to 0^-} \frac{f(-2 + h) - f(-2)}{h} = \lim_{h \to 0^-} \frac{b + 4}{h}
\]
\[
\lim_{h \to 0^+} \frac{f(-2 + h) - f(-2)}{h} = \lim_{h \to 0^+} \frac{(-2 + h)^2 + 4(-2 + h) + 4}{h} = \lim_{h \to 0^+} h = 0.
\]

For these limits to be equal, we need \(\lim_{h \to 0^-} \frac{b + 4}{h} = 0\), which is true for \(b = -4\).

**Question 5:**

(a) The line \(L\) tangent to \(y(x) = e^{x/3}\) at \(x = 0\) has slope \(y'(0)\). Now, \(y'(x) = \frac{1}{3} e^{x/3}\), so \(y'(0) = \frac{1}{3}\). Furthermore, \(y(0) = 1\), so \((0, 1)\) is a point on the tangent line. Since we found the slope of the tangent line and a point on it, we can now write its equation using point-slope form:
\[
L - 1 = \frac{1}{3} (x - 0) \implies L(x) = \frac{x}{3} + 1.
\]

(b) We know (see the Section 3.9 on linear approximation) that the tangent line approximates points on our function “near” the point of tangency. Thus, \(y(1) = e^{1/3}\) is approximated by \(L(1) = 1/3 + 1 = \frac{4}{3}\). To explain why \(\frac{4}{3} < e^{1/3}\), we need to explain why the tangent line \(L\) is an underestimate of the function \(y\). One can do this by simply noting the graph of \(y(x) = e^{x/3}\) is concave up. That is, since \(y\) is concave up, a line tangent at a point of \(y\) is strictly below the graph of \(y\), which is why \(L\) underestimates \(y\). Another way to see that \(y\) is concave up is to compute its second derivative, i.e.
\[
y''(x) = \frac{1}{9} e^{x/3} > 0 \implies y\text{ is concave up for all } x.
\]

**Question 6:**

\[
\frac{d}{dx} (\cos(xy)) = \frac{d}{dx} (y^2)
\]
\[
-\sin(xy) \cdot \frac{d}{dx} (xy) = 2y \cdot \frac{dy}{dx}
\]
\[
-\sin(xy) \left( y + x \cdot \frac{dy}{dx} \right) = 2y \cdot \frac{dy}{dx}
\]
\[
-2y \cdot \frac{dy}{dx} - x \sin(xy) \cdot \frac{dy}{dx} = y \sin(xy)
\]
\[
\frac{dy}{dx} = \frac{y \sin(xy)}{-2y - x \sin(xy)}
\]

**Question 7:**

To find the critical points of \(f(x)\), find \(f'(x)\), set it equal to zero, and solve for \(x\).
\[
f'(x) = 2xe^{-x} - x^2 e^{-x}
\]

Now, setting \(f'(x) = 0\) yields
\[
2xe^{-x} - x^2 e^{-x} = 0
\]
\[
e^{-x}(2x - x^2) = 0
\]
\[
2x - x^2 = 0
\]
\[
x(2 - x) = 0
\]

We found 2 critical points when \(x = 0\) and \(x = 2\). These points are \((0, f(0)) = (0, 0)\) and \((2, f(2)) = (2, 4e^{-2})\). To classify them, one can try using the First Derivative Test or the Second Derivative Test. We use the former. Looking at our work above, we can see that:
• For \( x < 0, f'(x) < 0 \).
• For \( 0 < x < 2, f'(x) > 0 \).
• For \( x > 2, f'(x) < 0 \).

Thus, since \( f'(x) \) goes from negative to positive at \( x = 0 \), \( f(x) \) has a local minimum of 0 at \( x = 0 \). And since \( f'(x) \) goes from positive to negative at \( x = 2 \), \( f(x) \) has a local maximum of \( 4e^{-2} \) at \( x = 2 \).

**Question 8:**

Let \( l \) denote the length of the plot (so, the vertical line in the center of the figure that divides the rectangle into two rectangles has length \( l \) ). Let \( w \) denote the width of the (entire) plot. Then, we have that the area of the whole plot is \( wl \), and the perimeter is \( 2w + 3l \). Since John has 1200 feet of fencing, this perimeter must equal 1200, i.e. \( 2w + 3l = 1200 \). So, \( w = 600 - \frac{3}{2}l \). Substituting this in for \( w \) in our expression for area, we obtain a function for area only depending on \( l \) which we want to optimize, i.e.

\[
A(l) = l(600 - \frac{3}{2}l)
\]

To find the maximum of this function, find \( A'(l) \), set this equal to zero, and solve for \( l \):

\[
A'(l) = 600 - 3l
\]

Setting this equal to zero and solving yields \( l = 200 \). Thus, a critical point occurs at \( l = 200 \). Since \( A''(l) = -3 \), by the Second Derivative Test, it must be that a local maximum occurs at \( l = 200 \). We still need to argue that this is a *global* maximum however. Luckily, this can be quickly seen by simply noting that \( A(l) \) is a parabola (that opens downward). Thus, the local maximum at \( l = 200 \) must be a global maximum.

We found that the dimensions maximizing area are:

\[
l = 200
\]
\[
w = 600 - \frac{3}{2} \cdot 200 = 300
\]