Definition of derivative: Let $f: \mathbb{R} \to \mathbb{R}$. $\frac{df}{dx}(x_0)$ is a number with the property that for any positive number $\varepsilon$, we can find a positive number $\delta$ so that if $0 < |\Delta x| < \delta$, then
\[
\left| \frac{\Delta f}{\Delta x}(x_0, \Delta x) - \frac{df}{dx}(x_0) \right| < \varepsilon.
\] Note that if no number has this property, we say that $f$ is not differentiable at $x_0$.

(1) Let $f(x) = 4x^2 + 2x$.

(a) Sketch the graph of $f$. Sketch the tangent line to the graph of $f$ at the point $(-1, f(-1))$.

(b) The slope of the tangent line to the graph of $f$ at the point $(-1, f(-1))$ is $-6$. This means that on a small interval around $-1$, the slopes of the secant lines through $(-1, f(-1))$ and $(-1 + \Delta x, f(-1 + \Delta x))$ should be close to $-6$. How small does $|\Delta x|$ need to be to guarantee that $\frac{\Delta f}{\Delta x}(-1, \Delta x)$ is within 0.01 of $-6$?

(c) How small does $|\Delta x|$ need to be to guarantee that $\frac{\Delta f}{\Delta x}(-1, \Delta x)$ is within 0.001 of $-6$?

(d) Use the definition of the derivative to show that $\frac{df}{dx}(-1) = -6$.

(e) Use the definition of the derivative to show that $\frac{df}{dx}(x_0) = 8x_0 + 2$.

(2) Let
\[
h(x) = \begin{cases} 
-1 & x \leq -4 \\
-x & x > -4 
\end{cases}
\]

(a) Sketch the graph of $h$. Sketch the tangent lines to the graph of $h$ at the points $(2, h(2))$ and $(-8, h(-8))$.

(b) Use the definition of the derivative to show that $\frac{dh}{dx}(x_0) = 0$ if $x_0 < -4$.

(c) Use the definition of the derivative to show that $\frac{dh}{dx}(x_0) = 1$ if $x_0 > -4$.

(d) Use the definition of the derivative to show that $\frac{dh}{dx}(-4)$ does not exist, i.e. show that $\frac{dh}{dx}(1) \neq A$ for any real number $A$.

(3) Let
\[
g(x) = \begin{cases} 
-5x & x \leq 2 \\
\frac{x}{2} - 11 & x > 2 
\end{cases}
\]

(a) Sketch the graph of $g$.

(b) Use the definition of the derivative to show that $\frac{dg}{dx}(x_0) = -5$ if $x_0 < 2$.

(c) Use the definition of the derivative to show that $\frac{dg}{dx}(x_0) = \frac{1}{2}$ if $x_0 > 2$.

(d) Use the definition of the derivative to show that $\frac{dg}{dx}(2)$ does not exist, i.e. show that $\frac{dg}{dx}(2) \neq A$ for any real number $A$. 

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