(1) Let 
\[ f(x) = \begin{cases} 
  x^2 \sin(1/x) & x \neq 0 \\
  0 & x = 0.
\end{cases} \]

Use the $\varepsilon - \delta$ definition of the derivative to show that $\frac{df}{dx}(0) = 0$.

(2) Use the $\varepsilon - \delta$ definition of the derivative to prove that the derivative of a sum equals the sum of the derivatives:

Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable functions with $f'(x_0) = A$ and $g'(x_0) = B$. Define a function $h : \mathbb{R} \to \mathbb{R}$ by the formula $h(x) = f(x) + g(x)$. Prove that $h'(x_0) = A + B$.

Hint: You will probably find useful the “triangle inequality” for absolute value, i.e. for all real numbers $a$ and $b$, $|a + b| \leq |a| + |b|$.

(3) Use the $\varepsilon - \delta$ definition of the derivative to prove:

If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and $f'(x_0) < 0$, then there exists some number $r > 0$ so that if $x \in (x_0, x_0 + r)$, then $f(x) < f(x_0)$, and if $x \in (x_0 - r, x_0)$, then $f(x) > f(x_0)$. 