121A: The Method Behind the Madness

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General Goals of the 121 Series:

Advancing your knowledge of mathematics
Advancing your knowledge of student learning (i.e. your view and understanding of the process of learning mathematics)
Advancing your knowledge of pedagogy (i.e. teaching practices)
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A Research-Based Framework for Teaching Mathematics: DNR-Based Instruction in Mathematics

The Math 121 series was designed in accordance with a theoretical framework called DNR. DNR aims at helping to:

- understand what it means to learn and teach mathematics
- make decisions as to what to teach and how to teach it

To learn more about DNR:
http://www.math.ucsd.edu/~harel
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What is DNR?

DNR-based instruction in mathematics is a conceptual framework that provides a language and tools to formulate and address critical curricular and instructional concerns. The term DNR refers to three foundational instructional principles:

- The Duality Principle
- The Necessity Principle
- The Repeated Reasoning Principle
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DNR aims to address

1. What is the mathematics that we should teach?
2. How can we teach it effectively?

In DNR, teaching effectively means:
- preserving the mathematical integrity of what we teach
- addressing the intellectual needs of the student
- assuring that students internalize and retain the mathematics they learn
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What is the Mathematics that We Should Teach?

Ways of Understanding/Subject Matter (Content):
definitions, problems and their solutions, algorithms, theorems, proofs, and so on

Ways of Thinking: conceptual tools necessary to develop understanding of subject matter, such as algebraic invariance, algorithmic reasoning, proportional reasoning, and deductive reasoning
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**The Duality Principle:**

Students acquire desirable ways of thinking by developing desirable understanding of content AND students’ current understanding of content is impacted by the ways of thinking they posses.

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Mental Acts

Human reasoning involves numerous mental acts such as interpreting, conjecturing, inferring, proving, explaining, generalizing, abstracting, predicting, classifying, and problem solving.

Mental acts are basic elements of human cognition. To describe, analyze, and communicate about humans intellectual activities, one must attend to their mental acts.
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Ways of Understanding Versus Ways of Thinking

A way of understanding is a cognitive product of a mental act.

A way of thinking is a cognitive characteristic of a mental act.
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Ways of Understanding Versus Ways of Thinking

- A **way of understanding** is a cognitive product of a mental act.
- A **way of thinking** is a cognitive characteristic of a mental act.
Consider the mental act of interpreting the string of symbols $y = 2x + 5$. 
Different ways of understanding the string of symbols $y = 2x + 5$:

- As an equation (a condition on the variables $x$ and $y$)
- As a number-valued function: for each number $x$, there corresponds the number $2x + 5$
- As a proposition-valued function: for every ordered pair $(x, y)$, there corresponds the value "true" or "false"

Ways of thinking manifested by these ways of understanding:

- Symbols in mathematics represent quantities and quantitative relationships
- Mathematical symbols can have multiple interpretations
  (would be manifested by one who exhibits more than one of the ways of understanding)
- It is advantageous to attribute different interpretations to a mathematical symbol in the process of solving problems
  (would be manifested by one who can vary the interpretation of the symbols according to the problem at hand)
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Example Two

Consider the mental act of problem solving.
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Examples of problem solving approaches:
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- Look for a simpler problem
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- Consider alternative possibilities while attempting to solve the problem
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Examples of problem solving approaches:

- Look for a simpler problem
- Consider alternative possibilities while attempting to solve the problem
- Just look for key words in the problem statement
Example Three

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Examples of proof schemes:

- Authoritative proof scheme: because the teacher says its true
- Empirical proof scheme: reliance on evidence from examples or visual perception
- Deductive proof scheme: one proves an assertion through a finite sequence of steps which follows from premises (and previous conclusions) through the application of rules of inference
What is the Mathematics that We Should Teach?

- **Ways of Understanding/Subject Matter (Content):** definitions, problems and their solutions, algorithms, theorems, proofs, and so on

**The Duality Principle:**
Students acquire desirable ways of thinking by developing desirable understanding of content AND students’ current understanding of content is impacted by the ways of thinking they posses.

- **Ways of Thinking:** conceptual tools necessary to develop understanding of subject matter, such as algebraic invariance, algorithmic reasoning, proportional reasoning, and deductive reasoning
Implications

Our question:

What is the mathematics that we should teach?

DNR's Answer:

Mathematical integrity: The mathematics curricula we teach must adhere to and maintain the essential nature of the mathematics discipline, focusing on both subject matter and ways of thinking.
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DNR's Answer: By addressing students' intellectual need.
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**DNR’s Answer:**

By addressing students’ intellectual need.
The Necessity Principle (The N in DNR)

The Necessity Principle:

For students to learn the mathematics we intend to teach them, they must have a need for it, where “need” refers to intellectual need, not social or economic need.
The Repeated Reasoning Principle (The R in DNR)

The Repeated Reasoning Principle:
Students must practice reasoning in order to organize, internalize, and retain what they learn.
Implications

Our question:

How can we teach mathematics effectively?

DNR's Answers:

1. Duality: We must attend to ways of thinking as well as ways of understanding.

2. Intellectual Need: Mathematics teaching must utilize humans remarkable capacity to be puzzled.

3. Internalization and retention: Repeated reasoning, not mere drill, must be the focus of teaching.
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Target Instructional Objectives

PGA way of thinking:
The ability to fluently connect the physical/perceptual aspects of a problem situation with the geometric aspects (e.g. graph) and the algebraic aspects (e.g. formulas and equations). One who possesses the PGA way of thinking searches for and exploits the correspondences between the physical, geometric, and algebraic aspects of a mathematical topic.
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Examples of Advancing the PGA Way of Thinking:

Alex runs up a hill and then back down to his starting point for a total distance of 12 kilometers. He runs 9 km/hr uphill and 16 km/hr downhill and uses the same path in both directions. What is Alex's distance from his starting point at any given moment from the time begins running?

Prove that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is a linear if and only if its average rate of change is the same on any interval.
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Target Instructional Objectives

Thinking in Terms of Functions as Processes and Models of Reality:

One who possesses this way of thinking understands a function as a dynamic transformation of quantities according to some repeatable means which, given the same original quantity, will always produce the same transformed quantity. In contrast, the most elementary conception of functions involves the ability to plug into an algebraic expression and calculate.
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Examples of Advancing this Way of Thinking:

Jack and Jill each run 10 kilometers. They start at the same point, run 5 kilometers up a hill, and return to the starting point by the same route. Jack has a 10 minute head start and runs at the rate of 10 km/hr uphill and 15 km/hr downhill. Jill runs 12 km/hr uphill and 18 km/hr downhill.

What is the distance between Jack and Jill at any given moment during the time they are both running?

You would like to predict the population of your town twenty years from now. How could you do this?

A spherical balloon is expanding. You want to determine the volume of the balloon at any given instant from the moment it started to expand. What do you do?
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Target Instructional Objectives

Deductive proof scheme:
The ability to produce deductive proofs and, in particular, the ability to conjecture, apply mental operations that are goal-oriented, and understand that all justification must be ultimately based on inference rules.
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Examples of Advancing the Deductive Proof Scheme:

Show that a sequence is a linear (aka arithmetic) sequence if and only if its sequence of first differences is a constant (non-zero) sequence.

Show that a sequence is a quadratic sequence if and only if its sequence of second differences is a constant (non-zero) sequence.

Show that a sequence is an exponential (aka geometric) sequence if and only if dividing any term in the sequence by the preceding term always results in the same (non-zero) number.
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Exportable Teaching Practices for Your Consideration

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