Math 121A: The Method Behind the Madness

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General Goals of the 121 Series:

- Advancing your knowledge of mathematics
- Advancing your knowledge of student learning (i.e., your view and understanding of the process of learning mathematics)
- Advancing your knowledge of pedagogy (i.e., teaching practices)
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The Math 121 series was designed in accordance with a theoretical framework called DNR. DNR aims at helping to:

- understand what it means to learn and teach mathematics
- make decisions as to what to teach and how to teach it

To learn more about DNR:
http://www.math.ucsd.edu/~harel
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What is DNR?

DNR-based instruction in mathematics is a conceptual framework that provides a language and tools to formulate and address critical curricular and instructional concerns. The term DNR refers to three foundational instructional principles:

- The Duality Principle
- The Necessity Principle
- The Repeated Reasoning Principle
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What is the mathematics that we should teach?

How can we teach it effectively?

In DNR, teaching effectively means:

- preserving the mathematical integrity of what we teach
- addressing the intellectual needs of the student
- assuring that students internalize and retain the mathematics they learn
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What is the Mathematics that We Should Teach?

Ways of Understanding/Subject Matter (Content):
- definitions, problems and their solutions, algorithms,
- theorems, proofs, and so on

Ways of Thinking: conceptual tools necessary to develop
understanding of subject matter, such as
- algebraic invariance,
- algorithmic reasoning,
- proportional reasoning,
- deductive reasoning
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The Duality Principle:

Students acquire desirable ways of thinking by developing desirable understanding of content AND students’ current understanding of content is impacted by the ways of thinking they possess.

- **Ways of Thinking:** conceptual tools necessary to develop understanding of subject matter, such as algebraic invariance, algorithmic reasoning, proportional reasoning, and deductive reasoning
Mental Acts

Human reasoning involves numerous mental acts such as interpreting, conjecturing, inferring, proving, explaining, generalizing, abstracting, predicting, classifying, and problem solving.

Mental acts are basic elements of human cognition. To describe, analyze, and communicate about humans intellectual activities, one must attend to their mental acts.
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Ways of Understanding Versus Ways of Thinking

A way of understanding is a cognitive product of a mental act.

A way of thinking is a cognitive characteristic of a mental act.
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Example One

Consider the mental act of interpreting the string of symbols $y = 2x + 5$. 
Different ways of understanding the string of symbols $y = 2x + 5$:

- As an equation (a condition on the variables $x$ and $y$)
- As a number-valued function: for each number $x$, there corresponds the number $2x + 5$
- As a proposition-valued function: for every ordered pair $(x, y)$, there corresponds the value "true" or "false"

Ways of thinking manifested by these ways of understanding:

Symbols in mathematics represent quantities and quantitative relationships.
Mathematical symbols can have multiple interpretations (would be manifested by one who exhibits more than one of the ways of understanding).
It is advantageous to attribute different interpretations to a mathematical symbol in the process of solving problems (would be manifested by one who can vary the interpretation of the symbols according to the problem at hand).
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Example Two

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- The actual solution that one provides to a problem, whether correct or erroneous, is a **way of understanding** because it is a particular cognitive product of the problem solving act.
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- Consider alternative possibilities while attempting to solve the problem
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Examples of problem solving approaches:

- Look for a simpler problem
- Consider alternative possibilities while attempting to solve the problem
- Just look for key words in the problem statement
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- A proof (a particular statement one offers to ascertain for oneself or to convince others) is a way of understanding because it is a particular cognitive product of the proving act.
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Example Three

Consider the mental act of **proving**.

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Examples of proof schemes:

- Authoritative proof scheme: because the teacher says its true
- Empirical proof scheme: reliance on evidence from examples or visual perception
- Deductive proof scheme: one proves an assertion through a finite sequence of steps which follows from premises (and previous conclusions) through the application of rules of inference
What is the Mathematics that We Should Teach?

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definitions, problems and their solutions, algorithms, theorems, proofs, and so on

**The Duality Principle:**
Students acquire desirable ways of thinking by developing desirable understanding of content AND students’ current understanding of content is impacted by the ways of thinking they posses.

**Ways of Thinking:** conceptual tools necessary to develop understanding of subject matter, such as algebraic invariance, algorithmic reasoning, proportional reasoning, and deductive reasoning
Target Instructional Objectives

PGA way of thinking: The ability to fluently connect the physical/perceptual aspects of a problem situation with the geometric aspects (e.g. graph) and the algebraic aspects (e.g. formulas and equations). One who possesses the PGA way of thinking searches for and exploits the correspondences between the physical, geometric, and algebraic aspects of a mathematical topic.
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Examples of Advancing the PGA Way of Thinking:

1. Alex runs up a hill and then back down to his starting point for a total distance of 12 kilometers. He runs 9 km/hr uphill and 16 km/hr downhill and uses the same path in both directions. What is Alex's distance from his starting point at any given moment from the time he begins running?

2. Prove that a function \( f: \mathbb{R} \rightarrow \mathbb{R} \) is linear if and only if its average rate of change is the same on any interval.

3. Let \( g(x) = |x| \).

   a. Sketch the graph of the function \( g \) and explain why the graph of \( g \) suggests that \( \frac{dg}{dx}(0) \) does not exist.

   b. Use the \( \varepsilon - \delta \) definition of the derivative to prove that \( \frac{df}{dx}(0) \) does not exist.
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  - algebraic proof

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  - geometric proof

- Let $g(x) = |x|$.
  1. Sketch the graph of the function $g$ and explain why the graph of $g$ suggests that $\frac{dg}{dx}(0)$ does not exist.
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Examples of Advancing the PGA Way of Thinking:

Let $f(x) = 3x^2 + 2x$.

1. Sketch the graph of $f$. Sketch the tangent line to the graph of $f$ at the point $(-1, 1)$.

2. The slope of the tangent line to the graph of $f$ at the point $(-1, 1)$ is $-4$. This means that on a small interval around $-1$, the slopes of the secant lines through $(-1, 1)$ and $(-1 + \Delta x, f(-1 + \Delta x))$ should be close to $-4$. How small does $|\Delta x|$ need to be to guarantee that $\Delta f/\Delta x(-1, \Delta x)$ is within 0.01 of $-4$?

Let $g(x) = x^2 - 5x + 7$. Verify that the conclusion of the Mean Value Theorem is true for $g$ on the interval $[-1, 3]$. Illustrate your answer with a sketch to demonstrate what is happening geometrically.
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Thinking in Terms of Functions as Processes and Models of Reality:

One who possesses this way of thinking understands a function as a dynamic transformation of quantities according to some repeatable means which, given the same original quantity, will always produce the same transformed quantity. In contrast, the most elementary conception of functions involves the ability to plug into an algebraic expression and calculate.
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Examples of Advancing this Way of Thinking:

Jack and Jill each run 10 kilometers. They start at the same point, run 5 kilometers up a hill, and return to the starting point by the same route. Jack has a 10 minute head start and runs at the rate of 10 km/hr uphill and 15 km/hr downhill. Jill runs 12 km/hr uphill and 18 km/hr downhill. What is the distance between Jack and Jill at any given moment during the time they are both running?

You would like to predict the population of your town twenty years from now. How could you do this?

A spherical balloon is expanding. You want to determine the volume of the balloon at any given instant from the moment it started to expand. What do you do?
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Deductive proof scheme:
The ability to produce deductive proofs and, in particular, the ability to conjecture, apply mental operations that are goal-oriented, and understand that all justification must be ultimately based on inference rules.
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Examples of Advancing the Deductive Proof Scheme:

Show that a sequence is a quadratic sequence if and only if its sequence of second differences is a constant (non-zero) sequence.

Let $f(x) = |x|$. Use the $\varepsilon-\delta$ definition of the derivative to prove that $\frac{df}{dx}(0)$ does not exist.

Prove Rolle's Theorem.

Prove that if two functions have the same derivative, then the functions differ by a constant.
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Definitional Reasoning:
A way of thinking by which one defines objects and proves assertions in terms of mathematical definitions. A mathematical definition is a description that applies to all objects to be defined and only to them. A crucial feature of this way of thinking is that, with it, one is compelled to conclude logically that there can be only one mathematical definition for a concept within a given theory; namely, if $D_1$ and $D_2$ are such definitions for a concept $C$, then $D_1$ is a logical consequence of $D_2$, and vice versa; otherwise, $C$ is not well defined.
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Examples of Advancing Definitional Reasoning:

Let \( f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases} \)

Use the \( \varepsilon - \delta \) definition of the derivative to show that \( \frac{df}{dx}(0) = 0 \).

Prove that if \( f : \mathbb{R} \to \mathbb{R} \) is a differentiable function and \( f'(x_0) > 0 \), then there exists some number \( r > 0 \) so that if \( x \in (x_0, x_0 + r) \), then \( f(x) > f(x_0) \), and if \( x \in (x_0 - r, x_0) \), then \( f(x) < f(x_0) \).

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DNR and the Standards

JMR-LS Conference
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1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
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Structuring lessons to allow repeated reasoning about concepts and ideas, to allow for internalization – a conceptual state where one is able to apply knowledge autonomously and spontaneously – and organization of knowledge.

Expecting the process of learning to often involve confusion, adjusting the trajectory of learning based on estimations of the learners' background knowledge.

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Exportable Teaching Practices for Your Consideration

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