Logical statements frequently come in the form “If $p$, then $q$”. Statements of this form are often also written as “$p$ implies $q$” and are thus called implications. For example, “If $n$ is odd, then $n^2$ is odd” is an implication and may also be written as “$n$ being odd implies that $n^2$ is odd”. Note that logical statements can either be true or false, but they cannot be both. The statement “If $n$ is odd, then $n^2$ is odd” is true (you proved that in HW2), but the statement “If $n$ is odd, then $n^2$ is even” is false.

Notation and vocabulary associated with implications:

1. The common notation for “If $p$, then $q$” is $p \Rightarrow q$.
2. The statement “If $q$, then $p$” is called the converse of “If $p$, then $q$”. For example, the converse of “If $n$ is odd, then $n^2$ is odd” is “If $n^2$ is odd, then $n$ is odd”. Note that the truth or falsity of the converse of an implication is completely independent of the truth or falsity of the implication itself. While “If $n$ is odd, then $n^2$ is odd” and its converse “If $n^2$ is odd, then $n$ is odd” are both true, the implication “If $n$ is even, then $2n$ is even” is true, but its converse “If $2n$ is even, then $n$ is even” is false.
3. The statement “$p$ and not $q$” is called the negation of “If $p$, then $q$”. For example, the negation of “If $n$ is odd, then $n^2$ is odd” is “$n$ is odd and $n^2$ is not odd”. Note that the negation of an implication is not an implication, but rather a so-called “and statement”. Note also that if an implication is true, its negation must be false. If an implication is false, its negation must be true.
4. The statement “If not $q$, then not $p$” is called the contrapositive of “If $p$, then $q$”. For example, the contrapositive of “If $n$ is odd, then $n^2$ is odd” is “If $n^2$ is not odd, then $n$ is not odd”, more succinctly expressed as “If $n^2$ is even, then $n$ is even”. Note that an implication and its contrapositive are logically equivalent, which means that they are either both true or both false.

Two applications to proofs:

1. Note that since an implication is true if and only if its contrapositive is true, if you are having difficulty proving that “$p$ implies $q$”, you can see if it is somehow easier to prove that “not $p$ implies not $q$”; this is called proof by contrapositive.
2. If you are having difficulty proving that “$p$ implies $q$”, you can also try proving that assuming its negation is true leads to a contradiction; specifically assume that “$p$ and not $q$” is true, and show that it leads to something absurd such as $0 = 1$. This technique is called proof by contradiction.

1They are also called conditional statements.