Math 121B, HW1

(1) Let $\overline{AB}$ be a segment. Let $l$ be the line through the midpoint of $\overline{AB}$ which is perpendicular to $\overline{AB}$. Prove:

(a) If $C$ is a point on the line $l$, then $AC = BC$. (Don’t forget to consider the case $C \in \overline{AB}$.)
(b) If $D$ is any point with the property that $AD = BD$, then $D$ is on the line $l$. (Don’t forget to consider the case $D \in \overline{AB}$.)

You have just proved that the two definitions of the perpendicular bisector of a segment are equivalent, i.e. you have proved that the perpendicular bisector of a segment is the set of all points which are equidistant from the endpoints of the segment!

(2) Suppose that $\overline{AB}$ and $\overline{BC}$ are chords of a circle. Let $m$ be the perpendicular bisector of $\overline{AB}$, and let $n$ be the perpendicular bisector of $\overline{BC}$. Prove that $m$ and $n$ intersect in the center of the circle. Note: This result is related to the second and third approaches to Circle Problem One.

(3) Suppose that $P$ and $Q$ are points on a circle and that $t$ and $s$ are the tangent lines to the circle at the points $P$ and $Q$, respectively. Let $l$ be the line through $P$ which is perpendicular to $t$, and let $m$ be the line through $Q$ which is perpendicular to $s$. Prove that if the lines $l$ and $m$ are not coincident (i.e. $t$ is not parallel to $s$), then $l$ and $m$ intersect in the center of the circle. Note: This result is related to the sixth approach to Circle Problem One.

(4) Suppose that a circle has been inscribed inside of a square (recall that this means that each of the sides of the square is tangent to the circle). Prove that the diagonals of the square intersect in the center of the circle. Note: This result is related to the first approach to Circle Problem One.

(5) All of our approaches to the Circle Problem One were based on the geometric definition of a circle, namely, a circle with center $O$ and radius $r$ is the set:

$$\{ P \text{ in the plane such that } OP = r \}$$

Note that in the above definition, $OP$ denotes the distance from $O$ to $P$, i.e. the length of the segment $\overline{OP}$. Use this geometric definition to derive an algebraic definition (i.e. equation) of a circle as follows. Introduce a coordinate system (an $x$-axis and a $y$-axis) on the plane so that the point $O$ has coordinates $(m, n)$. Find the equation satisfied by the points $(x, y)$ on the circle.

(6) Prove that a line and a circle intersect in two points or one point or not at all.