Computational tangent line problems: \textbf{Wherever applicable in these problems, please show the work that leads to your tangent line equation; don’t just use a formula from class.}

(a) Sketch the ellipse \( \frac{x^2}{3} + \frac{y^2}{2} = 1 \). Then find an equation of its tangent line at the point \((1, -1)\).

(b) Sketch the ellipse \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \). Then find an equation of its tangent line at the point \((-4, 0)\).

(c) Sketch the ellipse \( \frac{x^2}{4} + \frac{y^2}{4} = 1 \). Then find an equation of its tangent line at the point \((-\frac{8}{7}, -\frac{2}{7})\).

(d) Sketch the ellipse \( \frac{(x - \frac{11}{2})^2}{\frac{81}{4}} + \frac{(y - 1)^2}{\frac{81}{8}} = 1 \). Then find an equation of its tangent line at the point \((2, 3)\).

In the figure below, \(E\) and \(F\) are the foci of the ellipse and \(\overrightarrow{PR}\) is the line tangent to the ellipse at the point \(P\). In this problem you will prove that the tangent line creates equal angles with the focal radii, i.e. you will prove that \(\angle EPR \cong \angle FPQ\).

(a) To make it easier to discuss your solution with your classmates, let’s all use the same coordinate system. Let \(E = (-c, 0)\), \(F = (c, 0)\), and let the equation of the ellipse be \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\), where \(a^2 = b^2 + c^2\). Let the coordinates of the point \(P\) be \((x_0, y_0)\). To simplify your write-up, introduce the following notation for the angle measures. Let \(\gamma_1\) be the measure of \(\angle EPR\), and let \(\gamma_2\) be the measure of \(\angle FPQ\). So it is your job to prove that \(\gamma_1 = \gamma_2\). To do so, first show that \(\tan(\gamma_1) = \tan(\gamma_2)\). You can accomplish this using right triangles, some trig identities, and some algebra as follows. If we let \(\alpha\), \(\beta\), and \(\delta\) denote the measures of \(\angle PQF\), \(\angle PFS\), and \(\angle PES\), respectively, then \(\alpha + \gamma_2 = \beta\) (why?) and \(\alpha + \delta = \gamma_1\) (why?). So proving that \(\tan(\gamma_1) = \tan(\gamma_2)\) is equivalent to proving that \(\tan(\alpha + \delta) = \tan(\beta - \alpha)\). Using difference formulas, this means that you have to prove:

\[
\frac{\tan(\alpha) + \tan(\delta)}{1 - \tan(\alpha)\tan(\delta)} = \frac{\tan(\beta) - \tan(\alpha)}{1 + \tan(\beta)\tan(\alpha)}.
\]

To prove the above identity, use right triangles to compute the tangents of the relevant angles. In order to do so, you will need to use the equation of the tangent line to find the coordinates of the points \(Q\) and \(S\). Go for it!

(b) Generally speaking, if two angles have the same tangent, they need not have the same measure. In the context of this problem, however, \(\tan(\gamma_1) = \tan(\gamma_2)\) does indeed imply that \(\gamma_1 = \gamma_2\). Please explain why.
(3) Using the same notation as in the previous problem, here you will use an alternative method to prove that $\gamma_1 = \gamma_2$. In the figure below, $\overrightarrow{PK}$ is the line through $P$ perpendicular to the tangent line $\overrightarrow{PR}$.

(a) Show that $\overrightarrow{PK}$ is the angle bisector of $\angle EPF$. To do so, you may use the following theorem from geometry without reproving it:

**Theorem:** Let $\triangle EPF$ be as in the figure. Then $\overrightarrow{PK}$ is the angle bisector of $\angle EPF$ if and only if $\frac{EK}{EP} = \frac{FK}{FP}$.

To demonstrate that $\frac{EK}{EP} = \frac{FK}{FP}$, find the equation for the line $\overrightarrow{PK}$ and then use it to find the coordinates of the point $K$. You may also find useful the formulas for the lengths of the focal radii that we derived in lecture using Professor Rabin’s idea.

(b) Explain why the result from part (a) proves that $\gamma_1 = \gamma_2$.

(4) (a) Find an equation of the ellipse with foci $\left(\frac{5}{2}, 0\right)$ and $\left(\frac{13}{2}, 0\right)$ and one directrix the line $x = 10$.

(b) Find the eccentricity and the directrices (plural of directrix) of the ellipse

$$\frac{(x + 4)^2}{24} + \frac{(y - 1)^2}{49} = 1.$$ 

(c) Is there more than one ellipse with eccentricity $1/4$ and one directrix the line $x = 6$? If not, explain why not. If so, find the equations of two such ellipses.