**Problem One:** the figure below, $E$ and $F$ are the foci of the ellipse and $\overrightarrow{PR}$ is the line tangent to the ellipse at the point $P$. In this problem you will prove Xinyi’s conjecture that the tangent line creates equal angles with the focal radii, i.e. you will prove that $\angle EPR \cong \angle FPQ$.

To make it easier to discuss your solution with your classmates, let’s all use the same coordinate system. Let $E = (-c, 0)$, $F = (c, 0)$, and let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 = b^2 + c^2$. Let the coordinates of the point $P$ be $(x_0, y_0)$. To simplify your write-up, introduce the following notation for the angle measures. Let $\gamma_1$ be the measure of $\angle EPR$, and let $\gamma_2$ be the measure of $\angle FPQ$. So it is your job to prove that $\gamma_1 = \gamma_2$. To do so, first show that $\tan(\gamma_1) = \tan(\gamma_2)$. You can accomplish this using right triangles, some trig identities, and some algebra as follows. If we let $\alpha$, $\beta$, and $\delta$ denote the measures of $\angle PQF$, $\angle PFS$, and $\angle PES$, respectively, then $\alpha + \gamma_2 = \beta$ (why?) and $\alpha + \delta = \gamma_1$ (why?). So proving that $\tan(\gamma_1) = \tan(\gamma_2)$ is equivalent to proving that $\tan(\alpha + \delta) = \tan(\beta - \alpha)$. Using differences formulas, this means that you have to prove:

$$\frac{\tan(\alpha) + \tan(\delta)}{1 - \tan(\alpha) \tan(\delta)} = \frac{\tan(\beta) - \tan(\alpha)}{1 + \tan(\beta) \tan(\alpha)}.$$ 

To prove the above identity, use right triangles to compute the tangents of the relevant angles. In order to do so, you will need to use the equation of the tangent line to find the coordinates of the points $Q$ and $S$. Go for it!
Solution: WoT’s: Pause and strategize before making calculations. Observe and make use of structure/symmetry/anti-symmetry. Make clever/useful choices of notation to save time and trees. Use ALL of the relevant information, e.g. if \((x_0, y_0)\) is a point on the ellipse, then \(x_0\) and \(y_0\) satisfy the equation....

Using the tangent line equation, we find that the coordinates of the point \(Q\) are \(\left( \frac{a^2}{x_0}, 0 \right)\).

The coordinates of \(S\) are \((x_0, 0)\). Then we have:

\[
\tan(\alpha) = \frac{y_0 x_0}{a^2 - x_0^2} = \frac{y_0 x_0}{\frac{a^2}{y_0^2} y_0} = \frac{b^2 x_0}{a^2 y_0},
\]

\[
\tan(\beta) = \frac{y_0}{c - x_0}, \quad \text{and} \quad \tan(\delta) = \frac{y_0}{c + x_0}.
\]

So we need to show that

\[
1 - \left( \frac{b^2 x_0}{a^2 y_0} \right) \left( \frac{y_0}{c + x_0} \right) = 1 + \left( \frac{b^2 x_0}{a^2 y_0} \right) \left( \frac{y_0}{c - x_0} \right).
\]

The left hand side of the above equality is:

\[
\frac{(c+x_0)b^2x_0 + a^2y_0^2}{a^2y_0(c+x_0) - b^2x_0y_0} = \frac{a^2b^2 + b^2c_0}{a^2c_0y_0 + c^2x_0y_0},
\]

where the equality follows from canceling the denominators of the numerator and denominator and then using the identities \(b^2x_0^2 + a^2y_0^2 = a^2b^2\) and \(a^2 - b^2 = c^2\). Similarly, the right hand side of the equality that we need to show is:

\[
\frac{a^2y_0^2 - (c-x_0)b^2x_0}{a^2y_0(c-x_0) + b^2x_0y_0} = \frac{a^2b^2 - b^2c_0}{a^2c_0y_0 - c^2x_0y_0}.
\]

So we need to show that

\[
\frac{a^2b^2 + b^2c_0}{a^2c_0 + c^2x_0y_0} = \frac{a^2b^2 - b^2c_0}{a^2c_0 - c^2x_0y_0}.
\]

The “anti-symmetry” of the above equality looks quite interesting and special. In particular, if we let \(u = a^2b^2\), \(v = b^2c_0\), \(w = a^2c_0y_0\), and \(z = c^2x_0y_0\), then we are trying to show that

\[
\frac{u + v}{w + z} = \frac{u - v}{w - z},
\]

and cross-multiplication shows that this is equivalent to \(uz = vw\), which is easy to check!
**Problem Two:** Using the same notation as in the previous problem, here you will use an alternative method (Mike’s idea) to prove that $\gamma_1 = \gamma_2$. In the figure below, $\overrightarrow{PK}$ is the line through $P$ perpendicular to the tangent line $\overrightarrow{PR}$.

Show that $\overrightarrow{PK}$ is the angle bisector of $\angle EPF$. To do so, you may use the following theorem from geometry without reproving it:

**Theorem:** Let $\triangle EPF$ be as in the figure. Then $\overrightarrow{PK}$ is the angle bisector of $\angle EPF$ if and only if $\frac{EK}{EP} = \frac{FK}{FP}$.

To demonstrate that $\frac{EK}{EP} = \frac{FK}{FP}$, find the equation for the line $\overrightarrow{PK}$ and then use it to find the coordinates of the point $K$. You may also find useful the formulas for the lengths of the focal radii that we derived in lecture using Professor Rabin’s idea.

**Solution: WoT’s:** Pause and strategize before making calculations. Observe and make use of structure/symmetry/anti-symmetry. Make clever/useful choices of notation to save time and trees. Use ALL of the relevant information, e.g. if $(x_0, y_0)$ is a point on the ellipse, then $x_0$ and $y_0$ satisfy the equation....

The equation for the line $\overrightarrow{PK}$ is

$$y = \frac{a^2 y_0}{b^2 x_0} x - \frac{c^2 y_0}{b^2},$$

so the point $K$ has coordinates $\left(\frac{c^2 x_0}{a^2}, 0\right)$. Using our formulas for the lengths of the focal radii, we need to show that

$$\frac{a + \frac{c x_0}{a}}{\frac{c^2 x_0}{a^2} + c} = \frac{a - \frac{c x_0}{a}}{c - \frac{c^2 x_0}{a^2}}.$$

As in the previous solution, the “anti-symmetry” of the above equality is rather inviting. In particular, if we let $u = \frac{c x_0}{a}$ and $v = \frac{c^2 x_0}{a^2}$, then we are trying to show that

$$\frac{a + u}{c + v} = \frac{a - u}{c - v},$$

and cross-multiplication shows that this is equivalent to $av = uc$, which is easy to check!