**Theorem:** Suppose a plane and a cone are such that the plane does not contain the vertex of the cone and the axis of the cone is not orthogonal to the plane. If the angle between the slant plane and the axis of the cone is larger than the semi-angle of the cone, then the intersection of the plane and the cone is an ellipse.

**Proof using one Dandelin\(^1\) sphere:**

1. **Notation:** Let \( \delta \) be the semi-angle of the cone, let \( \Pi \) be the slant plane, let \( \alpha \) be the angle between \( \Pi \) and the axis of the cone, and let \( \rho \) be the intersection of \( \Pi \) and the cone. So we have assumed that \( \Pi \) does not contain the vertex of the cone and that \( \alpha > \delta \).
2. **Fit a sphere inside the cone so that it is tangent to both the plane \( \Pi \) and the cone.**
3. **Let \( F \) be the point of tangency of \( \Pi \) and the sphere.**
4. **The sphere intersects the cone in a circle; call the circle \( C \).**
5. **The circle \( C \) lies in a plane \( \Lambda \) which is orthogonal to the axis of the cone.**
6. **Let \( l \) be the line of intersection of the planes \( \Pi \) and \( \Lambda \).**
7. **We will show that \( \rho \) is an ellipse with focus \( F \) and directrix \( l \).** So we need to show that there is some number \( 0 < e < 1 \) such that for every \( P \in \rho \), we have \( \frac{PF}{\text{dist}(P,l)} = e \). Our idea will be to show that this ratio is independent of \( P \) and instead can be calculated using only the cone and the slant plane, and then we will justify why the ratio is strictly between zero and one.
8. **Let \( P \in \rho \), and let \( L \in l \) so that \( \overrightarrow{LP} \) is perpendicular to \( l \).**
9. **Extend the ray from the vertex \( V \) of the cone through \( P \) to meet \( C \) in \( Q \).**
10. **Since \( \overrightarrow{PF} \) and \( \overrightarrow{PQ} \) are both tangents from \( P \) to our sphere, \( PF = PQ \).**
11. **Let the vector \( \overrightarrow{LM} \) be the orthogonal projection of the vector \( \overrightarrow{LP} \) on the plane \( \Lambda \), i.e. \( \overrightarrow{MP} \) is a normal vector to \( \Lambda \).**
12. **Since \( \angle PMQ \) and \( \angle PML \) are both right angles, we have**
    \[
    PQ = \frac{PM}{\sin \beta} \quad \text{and} \quad PL = \frac{PM}{\sin \gamma},
    \]
    where \( \beta \) is the measure of \( \angle PQM \) and \( \gamma \) is the measure of \( \angle PLM \).
13. **Thus,**
    \[
    \frac{PF}{\text{dist}(P,l)} = \frac{PF}{PL} = \frac{PQ}{PL} = \frac{\sin \gamma}{\sin \beta}.
    \]
14. **But \( \beta \) is the base angle of the cone, so \( \beta = \pi/2 - \delta \). And \( \gamma \) is the angle between the planes \( \Pi \) and \( \Lambda \), i.e. \( \gamma \) is the angle between normal vectors to \( \Pi \) and \( \Lambda \), which means that \( \gamma = \pi/2 - \alpha \). By assumption, \( \delta < \alpha \), so \( \gamma < \beta \). Together with \( \gamma, \beta \in (0, \pi/2) \), this implies that \( 0 < \frac{\sin \gamma}{\sin \beta} < 1 \).**
15. **Note that \( \beta \) only depends on the cone, and \( \gamma \) only depends on the slant plane and the cone. If we had started with a different point \( B \in \rho \), we would still have**
    \[
    \frac{BF}{\text{dist}(B,l)} = \frac{\sin \gamma}{\sin \beta}. \quad \text{So} \ \rho \ \text{is an ellipse with focus} \ F, \ \text{directrix} \ l, \ \text{and eccentricity} \ \frac{\sin \gamma}{\sin \beta},
    \]

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\(^1\)Germinal Pierre Dandelin (1794-1847) was a Belgian Professor of Mechanics at Liege University; he made his discovery in 1822.