Hyperbola Definitions, Math 121B

Fix points $F_1 = (-c,0)$ and $F_2 = (c,0)$, where $c > 0$. Fix a number $a$ such that $0 < a < c$. Then the hyperbola with foci $F_1$ and $F_2$ and vertices $(-a,0)$ and $(a,0)$ has four equivalent definitions:

1. $\left\{ \text{points } P \in \mathbb{R}^2 \text{ such that } |PF_1 - PF_2| = 2a \right\}$

2. $\left\{ (x, y) \in \mathbb{R}^2 \text{ such that } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } c^2 = a^2 + b^2 \right\}$

3. $\left\{ \text{points } P \in \mathbb{R}^2 \text{ such that } PF_1 = e \cdot \text{dist}(P, l_1), \text{ where } e = \frac{c}{a} \text{ and } l_1 \text{ is the line } x = -\frac{a^2}{c} \right\}$

4. $\left\{ \text{points } P \in \mathbb{R}^2 \text{ such that } PF_2 = e \cdot \text{dist}(P, l_2), \text{ where } e = \frac{c}{a} \text{ and } l_2 \text{ is the line } x = \frac{a^2}{c} \right\}$

Note that the number $e$ is called the eccentricity of the hyperbola, and the lines $l_1$ and $l_2$ are called the directrices of the hyperbola.