Math 18 Written Homework Three
Note that this homework is two pages long!

1. Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the transformation defined by \( T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \). Let \( \vec{u} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} \), and let \( \vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \).

Find \( T(\vec{u}) \) and \( T(\vec{v}) \) and then sketch the four vectors \( \vec{u}, \vec{v}, T(\vec{u}), \) and \( T(\vec{v}) \).

2. Suppose vectors \( \vec{u}_1, \ldots, \vec{u}_q \) span \( \mathbb{R}^m \) and suppose that \( S : \mathbb{R}^m \to \mathbb{R}^m \) is a linear transformation with \( S(\vec{u}_i) = \vec{0} \) for each \( i = 1, \ldots, q \). Show that \( S \) is the zero transformation, i.e. show that for any \( \vec{x} \) in \( \mathbb{R}^m \), \( S(\vec{x}) = \vec{0} \).

3. Suppose \( T : \mathbb{R}^4 \to \mathbb{R}^5 \) is a one-to-one linear transformation. List all of the possible echelon forms of the standard matrix for \( T \). Use the notation of Example 1 in Section 1.2 of your textbook (page 13 of the hard copy of the book). In particular, ■ denotes a nonzero entry, * denotes an entry that can be either zero or non-zero, and 0 denotes an entry that must be zero. For example, using this notation, one possible echelon form of a \( 4 \times 4 \) matrix is:

\[
\begin{bmatrix}
■ & * & * \\
0 & ■ & * \\
0 & 0 & ■ \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

This isn’t one of the answers for this problem, because the dimensions are wrong, but your answer should be a list of matrices of this nature.
4. Suppose \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) is an onto linear transformation. List all of the possible echelon forms of the standard matrix for \( T \). Use the notation of problem three above.

5. If a linear transformation \( T : \mathbb{R}^a \rightarrow \mathbb{R}^b \) is one-to-one, what must be true about \( a \) and \( b \)? i.e. can you give a relation between \( a \) and \( b \)? Please explain the reason for your answer using complete sentences.

6. If a linear transformation \( T : \mathbb{R}^a \rightarrow \mathbb{R}^b \) is onto, what must be true about \( a \) and \( b \)? i.e. can you give a relation between \( a \) and \( b \)? Please explain the reason for your answer using complete sentences.