Math 18 Written Homework Seven
Note that this homework is three pages long!

1. If the given statement is true, write “True”. If the given statement is false, write “False” and explain why it is false using complete sentences with proper grammar and punctuation. Giving an example is often a great way to demonstrate that a statement is false!

(a) If $B$ is any echelon form of a given matrix $A$, then the pivot columns of $B$ form a basis for the column space of $A$.

(b) Row operations preserve the linear dependence relations among the rows of a matrix. In particular, the linear dependence relations among the rows of a matrix are exactly the same as the linear dependence relations among the rows of the RREF of the matrix.

(c) The dimension of the null space of a matrix $A$ equals the number of columns of $A$ that are not pivot columns.

(d) The row space of a matrix $A$ is the same as the column space of the matrix $A^T$.

(e) If $A$ and $B$ are row equivalent matrices, then the row space of $A$ is the same as the row space of $B$. 
2. Suppose $A$ is a $7 \times 9$ matrix. What is the smallest possible dimension of $\text{Nul } A$? What is the largest possible dimension of $\text{Nul } A$?

3. Suppose a nonhomogeneous system of eight linear equations in ten unknowns has a solution, with two free variables. Is it possible to change some constants on the right hand sides of the equations to create a new inconsistent system? Please make sure to explain the reason for your answer.

4. Suppose a nonhomogeneous system of eleven linear equations in twelve unknowns has a solution for all possible constants on the right hand sides of the equations. Is it possible to find two nonzero solutions of the associated homogeneous system that are not multiples of one another? Please make sure to explain the reason for your answer.

5. A scientist solves a nonhomogeneous system of eighteen linear equations in twenty unknowns and finds that three of the unknowns are free variables. Can the scientist be certain that, if the right sides of the equations are changed, then the new nonhomogeneous system will still have a solution? Please make sure to explain the reason for your answer.
6. Let \( S \) be the set of polynomials \( \{1 - 2t + t^2, \ t - 2t^2 + t^3, \ 1 - 3t + 3t^2 - t^3\} \). Determine whether or not \( S \) is a linearly independent set. **Hint:** Start by finding the coordinate vectors of the polynomials; remember that the degree three polynomial \( p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \) has coordinate vector \( \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \). Then use the fact the polynomials in the set are linearly independent if and only if their coordinate vectors are linearly independent.

7. The first four Hermite polynomials are the set \( \{1, \ 2t, \ -2 + 4t^2, \ -12t + 8t^3\} \). These polynomials arise naturally in the study of certain important differential equations in mathematical physics. Show that the first four Hermite polynomials form a basis of \( \mathbb{P}_3 \). **Hint:** Start by finding the coordinate vectors of the polynomials; remember that the degree three polynomial \( p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \) has coordinate vector \( \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \). Then use the fact the polynomials form a basis of \( \mathbb{P}_3 \) if and only if their coordinate vectors form a basis of \( \mathbb{R}^4 \).