Math 18 Review Problems.
Disclaimer: These problems are not written based on the content or form of your final exam. They are just good practice with the material!

1. Find the best fit line for the data points $(-3, 0), (-2, 7), (1, 20), (4, 36)$.

2. For this entire problem let $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 12 \\ -1 \\ 2 \end{bmatrix}$.
   (a) Show that the set $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set.
   (b) Let $W = \text{span}\{\vec{u}_1, \vec{u}_2\}$. Find the point in $W$ which is closest to $\vec{x}$.

3. Let $A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix}$.
   (a) Find all of the eigenvalues of $A$. Please show your work.
   (b) For each eigenvalue from part (a), find a basis for the corresponding eigenspace. Please show your work.
   (c) Is $A$ diagonalizable? Please write “Yes” or “No”.
   (d) If you wrote “Yes” for part (c), find a diagonal matrix $D$ and an invertible matrix $P$ such that $A = PDP^{-1}$. If you wrote “No” for part (c), explain why $A$ is not diagonalizable.

4. Suppose that $A = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ and $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ are bases for a three dimensional vector space $V$, and suppose also that $\vec{a}_1 = \vec{b}_1 - \vec{b}_2$, $\vec{a}_2 = \vec{b}_1 + \vec{b}_2 - \vec{b}_3$, $\vec{a}_3 = 2\vec{b}_2 + 2\vec{b}_3$.
   (a) Find the change of coordinate matrix from $A$ to $B$.
   (b) If $\vec{x}$ is vector in $V$ with $[\vec{x}]_A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $[\vec{x}]_B$.

5. (a) Suppose that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6$. Find $\begin{vmatrix} 3d+a & 3e+b & 3f+c \\ g & h & i \end{vmatrix}$.
   (b) Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(3, 2, 1)$, $(2, 2, 5)$, and $(-1, 3, 0)$.

6. (a) Let $H$ be the set $\{p \in \mathbb{P}_3$ with the property that $p(0) = 0\}$. Is $H$ a subspace of the vector space $\mathbb{P}_3$? Carefully justify your answer.
   (b) Let $K$ be the set of points in $\mathbb{R}^2$ which are either on or inside the circle of radius one with center at the origin, i.e. $K = \{(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 \leq 1\}$. Is $K$ a subspace of $\mathbb{R}^2$? Carefully justify your answer.

7. Let $p_1(t) = 1 - t + 2t^2$, $p_2(t) = -2 + 3t - 5t^2$, $p_3(t) = 3 + 5t^2$. The set $\{p_1(t), p_2(t), p_3(t)\}$ is a basis for $\mathbb{P}_2$; you do not have to show this. Your job: Write the polynomial $q(t) = 9 - 4t + 17t^2$ as a linear combination of $p_1(t), p_2(t), \text{ and } p_3(t)$.
8. Let \( B = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \).

The reduced row echelon form of \( B \) is \( \text{rref}(B) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \).

(a) List three nonzero vectors in the null space of \( B \).
(b) Find a basis for the row space of \( B \). What is the dimension of the row space of \( A \)?
(c) Is the equation \( B\vec{x} = \vec{d} \) consistent for every choice of \( \vec{d} \in \mathbb{R}^4 \)? Please write “Yes” or “No” and explain the reason for your answer using complete sentences.
(d) Show how to write one of the columns of \( B \) as a linear combination of the other columns of \( B \). In your answer, please use the notation \( \vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5 \) to denote the five columns of \( B \).

9. (a) Is there a linear transformation \( S: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that
\[
S \left( \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad S \left( \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad S \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 8 \end{bmatrix}.
\]
If so, give an example of such an \( S \); if not, explain why not.
(b) Let \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^4 \) be the linear transformation such that
\[
T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -4 \\ 8 \\ 12 \\ 4 \end{bmatrix} \quad \text{and} \quad T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 8 \\ 2 \\ 3 \\ 5 \end{bmatrix}.
\]
Find the standard matrix of \( T \). Is \( T \) one-to-one? Is \( T \) onto?

10. (a) Suppose a nonhomogeneous system of 15 linear equations in 17 unknowns has a solution for all possible constants on the right hand sides of the equations. If \( \vec{v} \) and \( \vec{w} \) are solutions of the associated homogeneous system, must they line on the same line? Please write “Yes” or “No” and explain the reason for your answer using complete sentences.
(b) Suppose \( A \) is a square matrix and that \( \vec{v} \) is an eigenvector of \( A \). Show that \( \vec{v} \) is also an eigenvector of the matrix \( A^2 \).