Math 18, Handout One

**Vocabulary**: An \( m \times n \) matrix is a matrix with \( m \) rows and \( n \) columns.

**Elementary row operations\(^1\):**

1. **Replacement**: Replace one row by the sum of itself and a multiple of another row
2. **Interchange**: Interchange two rows
3. **Scaling**: Multiply all entries in a row by a nonzero constant

**Vocabulary**: Two matrices are **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other. If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

A **reduced row echelon form** (RREF) matrix is a matrix with the following properties:

1. If the matrix has any rows of zeros, they are below the rows with at least one nonzero entry.
2. The leftmost nonzero entry in a row is 1 (called a **leading one**).
3. If two rows each contain a leading one, then the column containing the leading one of the higher row is to the left of the column containing the leading one of the lower row.
4. All columns containing leading ones have no other nonzero entries.

Note: A matrix with properties (1) - (3) is said to be in **echelon form**. A given matrix can have many echelon forms, but its RREF is unique.

**Vocabulary**:

1. A **pivot position** in a matrix \( A \) is a location in \( A \) that corresponds to a leading one in the RREF of \( A \).
2. A **pivot column** of a matrix \( A \) is a column of \( A \) that contains a pivot position.
3. A linear system is called **consistent** if it has at least one solution.
4. A linear system is called **inconsistent** if it does not have any solution.
5. Given a matrix for a linear system, the variables that correspond to the pivot columns of the matrix are called **basic variables**; the variables that do not correspond to the pivot columns of the matrix are called **free variables**.

**Gauss-Jordan Elimination Algorithm** to find the RREF of a matrix:

1. **Step Zero**: If necessary, interchange rows until all rows of zeros are at the bottom of the matrix.
2. **Step One**: Identify the first pivot column; it is the leftmost column with at least one non-zero entry. Use row operations to obtain a one in its top position.
3. **Step Two**: Use row operations to obtain zeros below the leading one in the first pivot column.
4. **Step Three**: Consider only the columns to the right of the first pivot column and the rows below the first row. Repeat steps one and two on this submatrix.
5. **Step Four**: Continuing to work from left to right, repeat step three until your matrix has leading ones in all pivot positions and zeros below all of these leading ones. Now your matrix is in an echelon form.
6. **Step Five**: Find the rightmost leading one. Working from bottom to top, use row operations to obtain zeros in the entries directly above it. Working from right to left, repeat this process until the matrix is in RREF.

**Theorem**: A linear system is consistent if and only if the last column of its augmented matrix is not a pivot column (i.e. the RREF has no row of the form \([0 \cdots 0 b]\), where \( b \neq 0 \)). If the system is consistent, then the system has a unique solution if every column but the last is a pivot column, or it has infinitely many solutions if there are at least two non-pivot columns (including the last).

\(^1\)All of the elementary row operations are reversible, i.e. if you transform a matrix using an elementary row operation, you can use a corresponding elementary row operation to get back to your original matrix.