1. (a) Is there a linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ such that
\[
S \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad S \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad S \left( \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\]
If so, give an example of such an $S$; if not, explain why not.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the linear transformation such that
\[
T \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}.
\]
Find the standard matrix of $T$.

2. Let $P_2$ denote the vector space of polynomials of degree $\leq 2$
\[
P_2 = \{ a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \in \mathbb{R} \}.
\]
Let $W$ be the subspace of $P_2$ spanned by $\{1 - t, 1 - t^2, t^2 - t\}$. Find a basis $B$ for $W$. What is the dimension of $W$?

3. For this entire problem, let $A = \left\{ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 10 \\ -6 \\ 8 \\ 0 \end{bmatrix} \right\}$. Note that $A$ is a basis for $\mathbb{R}^4$.

(a) Suppose $[\bar{x}]_A = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$. Find $\bar{x}$.

(b) Let $B = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\}$ Note that $B$ is a basis for $\mathbb{R}^4$. Find the change of coordinate matrix from $A$ to $B$.

4. (a) Let $A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$ Find the eigenvalues of $A$ as well as the corresponding eigenspaces.

(b) Let $A$ be the matrix from part (a). Find a diagonal matrix $D$ and an invertible matrix $P$ so that $A = PDP^{-1}$. You don’t have to calculate $P^{-1}$.

5. (a) Let $H$ be the set of all matrices of the form $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$, with the property that $a + e + k = 0$. Is $H$ a subspace of the space of all $3 \times 3$ matrices? Please justify your answer.

(b) Let $K$ be the set of all $2 \times 2$ matrices with determinant equal to 0. Is $K$ a subspace of the set of all $2 \times 2$ matrices? Please justify your answer.

6. (a) Give an example of a set $S$ which spans $\mathbb{R}^3$ but is not a basis for $\mathbb{R}^3$. Briefly explain why your set $S$ is not a basis.

(b) Suppose that $M$ is a $5 \times 8$ matrix. Can the null space of $M$ be two-dimensional? Please explain.

(c) Suppose that $A$ is a $4 \times 6$ matrix and that $Ax = b$ is consistent for every $b$ in $\mathbb{R}^4$. Is $A^T x = c$ consistent for every $c$ in $\mathbb{R}^6$? Please explain.

(d) Suppose $B$ is a square matrix and that $v$ and $w$ are eigenvectors of $B$ with the eigenvalue of $v$ equalling 6 and the eigenvalue of $w$ equalling 5. Show that $v$ and $w$ must be linearly independent.